GEOMETRICAL LECTURES

ISAAC BARROW

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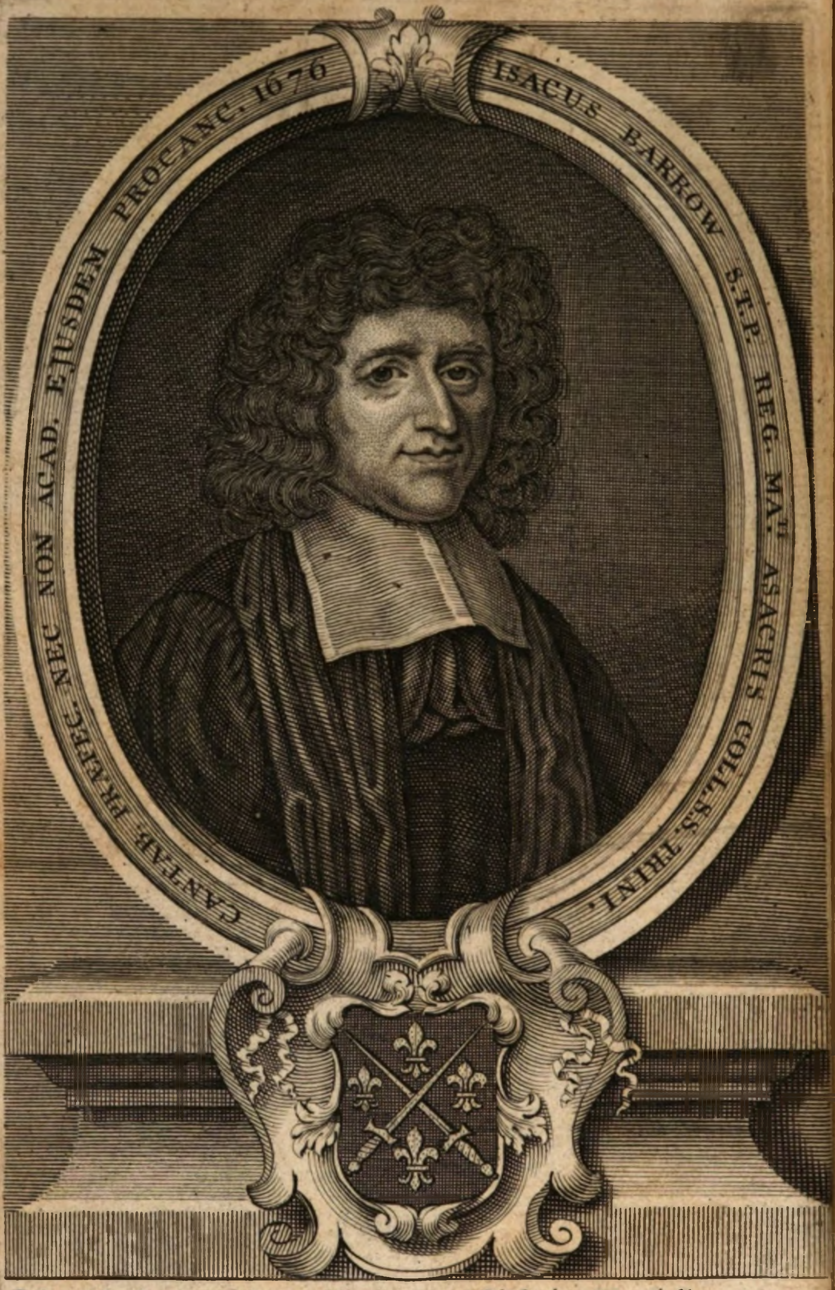
ISACUS BARROW S.T.P. REG. MA^TI. A SACRIS COLL. S.S. TRINI. CANTAB. PRÆFEC. NEC NON ACAD. EJUSDEM PROCANC. 1676

Printed for Stephen Auſten at the Angel and Bible in St. Paul's Church Yard

UNIV. OF H.

GEOMETRICAL LECTURES:

Explaining the
GENERATION, NATURE and PROPERTIES
OF
CURVE LINES.

Read in the
UNIVERSITY of CAMBRIDGE,

BY
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AND
Maſter of *Trinity-College*, &c.

RADCLIFFE
OBSERVATORY
OXFORD.

Tranſlated from the *Latin* Edition, reviſed, corrected and amended by the late Sir *Iſaac Newton*.

BY
EDMUND STONE, F. R. S.

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THE AUTHOR's PREFACE

TO HIS OPTICK and GEOMETRICAL LECTURES.

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COURTEOUS READER,

THIS ſmall Work, ſuch as it is, you will upon peruſal immediately perceive by many Tokens, not to have been deſigned to be made

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Publick, tho' I had often been importuned by ſeveral ſo to do. With the repeated Inſtances of whom, but not without Fear and Reluctance, I was at length prevailed upon; chiefly becauſe it ſeem'd to me a laudable Ambition, and even incumbent upon me as Part of my Duty, to endeavour, if not effectually to promote Learning, by ſetting an Example to thoſe who ſhould ſucceed me in the Profeſſorſhip, with which I am the firſt that was honoured. And likewiſe becauſe I have a ſmall Opinion of its containing ſomething that will both profit and pleaſe. Notwithſtanding I deſire you, who are well verſed in Matters of this Kind to remember what thoſe Things are which you are peruſing; not ſuch as were deſigned for you alone; not publiſhed of my own accord; nor containing the exquiſite Thoughts of a Mind wholly employ'd upon the Subject;

but that they are School Lectures, read by Duty, and ſometimes ſpoken too haſtily, that I might end my deſigned Taſk within the Hour; and laſtly, that they were accommodated for the Inſtruction of a promiſcuous Multitude of Scholars, to whom I was obliged to explain many Things which to you need not have been done. All this you are to remember, that you ſhould not expect here any Thing accurately performed, or beautifully digected. For if ſuch were your Expectations you would be diſſatisfied both with your ſelf and me. For I know that to your Satisfaction, I ought to curtail many Things, and ſubſtitute better in their room; to tranſpoſe the greater Part, and to correct and reform the Whole: Which to attempt, I neither had Inclination or Leiſure, no nor Power to execute. I therefore choſe rather to publiſh them in puris Naturalibus, *or as they were produced at firſt, than be at the Trouble of reducing them into any other Form, which perhaps might have diſpleaſed. For when I reſolv'd to publiſh, I could not bear the Pains of readding over again a great Part of theſe Things; either from my being tired with them, or not caring to undergo the Pains and Study in new modelling them: But I have done in this as weakly Mothers, who give up their Offspring to the Care of their Friends, either to Nurſe and bring up, or abandon to the wide World. One of which (for I think my ſelf bound to Name them) is Mr.* Iſaac Newton, *my Collegue, a Man of great Learning and Sagacity, who reviſed my Copy and noted ſuch Things as wanted Correction, and even gave me ſome of his own, which you will ſee here and there interſperſed with mine, not without*

without their due Commendations. The other is Mr. John Collins *(who may be deſervedly called the* Merſennas *of our Nation, Born to promote this Science, both with his own Labours, and thoſe of others. Who with much Trouble took care of the Edition. I could alſo put ſeveral other Obſtacles to your Expectations, or offer many other Excuſes* (viz. *the Narrowneſs of my own Genius, the Want of Experiments, and Buſineſs interfering) did not I fear* Cato *the Senior's Jeſt, ſhould be now thrown at me,* Recte ſi Amphictyonum decreto conſtrictus hæc evulgas. *Juſtice partly required a Preface of this Kind from me, and I was partly induced to it by the natural Love of a Parent for his own Offspring, that this Production might the more readily be excuſed, and in ſome Meaſure preſerved from Cenſure. But if you are*

are too ſevere and will not admit of theſe Excuſes, you may (for me) blame on.

ERRATA.

Page 16. line 23. read *endued.* p. 20. l. 30. r. *this.* p. 27. l. 14. r. *bearing.* p. 32. l. 19. dele *right.* p. 33. l. 1. r. *moving.* p. 47. l. 23. r. *Peripheries.* p. 64. l. 18. r. *the.* p. 74. l. 26. r. *thus.* p. 80. l. 25. *for ſo is* TP) r. TP *Cub.* p. 96. l. 2. r. *if the.* p. 113. r. *Fig.* 53. p. 118. l. 4. r. *Semi-Axes.* p. 153. l. 8. r. 3 SP. p. 158. l. 17. dele *right.* p. 174. l. 6. r. *Tangent.* ibidem l. 23. r. AQ. p. 223. r. *give us ſomething elſe.* p. 226. l. 4. r. *ſame things.*

GEOMETRICAL LECTURES.

LECTURE I.

I AM now entring into a new Field, whether more pleaſant or fruitful, I cannot truly ſay, but yielding a moſt copious Variety which conſequently is agreeable; and as it comprehends, for the moſt Part, the *Original of Mathematical Hypotheſes*, from whence Definitions are formed, and Properties flow, it muſt Neceſſarily be very uſeful too. What I mean, is the Genera-

tion of Magnitudes, or the ſeveral Ways whereby the various Species of Magnitudes may be conceiv'd to be generated or produced. Nor indeed is there any Magnitude given, but what may be conceiv'd to be produced, and really is produced innumerable Ways; yet theſe may be brought under ſome general Heads, at leaſt ſuch as have hitherto been uſed, of which the following occur to me at this Time; by *local Motion*; by the *Interſections of Magnitudes*; *by determinate Diſtances from aſſigned Places in Quantity and Poſition*; by the drawing *of Magnitudes into Magnitudes*; and by the *Applications of Magnitudes* to Magnitudes; by the *joining* together of Magnitudes peculiarly diſpoſed; by *the Appoſition or adding of Magnitudes* to other Magnitudes, *or by their Subduction* from others; or laſtly *by organical Effection* (deduced or contrived from any one of theſe.) Among theſe Ways, or any other whatever, of generating Magnitudes, the Primary and Chief is, that perform'd *by local Motion*, which all of them muſt in ſome Sort ſuppoſe, becauſe without Motion, nothing can be generated or produced, and therefore this muſt firſt be conſidered. The following Axiom of *Ariſtotle* concerning Motion is famous, ἀναγκαῖον ἀγνοουμένης
3 Phyſ. αὐτῆς (κινήσεως) ἀγνοεῖσθαι καὶ τὴν φύσιν: *He that is ignorant of Motion, muſt neceſſarily know nothing of Nature*; and conſequent-

ly nothing can be done without it in Phyſicks. Nor is this aſſerted without Reaſon, becauſe that in Nature, (ſo far at leaſt as our Underſtandings can trace, or Experience ſhew) every Thing created is produc'd by Motion, or certainly not without Motion. Accordingly *Philoſophers* argue with great Subtilty on the Nature of Motion, its true Definition; the Cauſes thereof, and the ſeveral Differences, all which are of very little Concern to *Mathematicians.* 'Tis enough that theſe take for granted what is allow'd by common Senſe, and proved by obvious Experience; and firſt, this general Principle, *viz.* That any Magnitude (among which I eſteem even a Point, as the leaſt of all Magnitudes, and likewiſe an Infinite or greateſt Magnitude, between which two, all intermediate finite Magnitudes are plac'd) is moveable; that is, in what Manner ſoever we behold it, the ſame may be made to change its Place continually, according to preſcribed Differences, *viz.* with a ſtrait or circular Motion; equally Swift, or in any manner more accelerated, or retarded: I ſay, *Mathematicians* aſſume at Pleaſure any Motions of this Kind as evidently poſſible, in order to find out and demonſtrate what follows from thence. We therefore ſhall treat of theſe Differences of Motion, their Number and Qualities. What *Mathematicians* chiefly conſider in Motion, is, the

Mode of Lation or *Manner of bearing, and the Quantity of the motive Force:* And firſt, the Mode of Lation, according to which, ſome Motions are Progreſſive, others Circumrotatory, and others made up of both theſe; then, the Quantity of the motive Force, according to which, One is declar'd ſwifter, ſlower, or equally ſwift in Reſpect of another; or ſaid to be equal in itſelf, accelerated or retarded. From theſe Springs the Differences of Motions flow, the latter whereof we ſhall treat firſt, becauſe it contains ſome foreign Things which I would firſt remove, in order that thoſe remaining may afterwards be more perſpicuous. But becauſe the Quantity of motive Force cannot be known without Time, we muſt ſay ſomething concerning its Nature. Now pray tell me what Time is? You know the very trite Saying of St. *Auſtin*, *If no one aſks me, I know; but if any Perſon ſhould require me to tell him, I cannot.* But becauſe *Mathematicians* frequently make uſe of Time, they ought to have a diſtinct Idea of the meaning of that Word, otherwiſe they are Quacks. My Auditors may therefore, on this Occaſion, very juſtly require an Anſwer from me, which I ſhall now give, and that in the plaineſt and leaſt ambiguous Expreſſions, avoiding as much as poſſible all trifling and empty Words. *Time*, (to ſpeak abſtractedly) is the continuance of any Thing in its own Being. But ſome Things continue longer in their Beings than others; thoſe *were* when theſe were not, and *are* when theſe are not; they enter'd firſt into Being, and ceaſe to be after theſe; nor is there any Perſon but perceives, that ſome Things enter into Being, and ceaſe to be at the ſame Time; keeping an equal Pace, as it were, from the beginning to the end of their Duration. Time abſolutely therefore is Quantity, as admitting in ſome Manner the chief Affections of Quantity, Equality, Inequality, and Proportion; nor do I believe there is any One but allows that thoſe Things exiſted equal Times, which roſe and periſhed together; and that thoſe Things had unequal Durations, when the one was in Being before the other had exiſtance, and continue in its Being, after the other had ceaſed to be. But a longer and ſhorter Time is common in every Body's Mouth; and there is no Man but ſeems to underſtand the Meaning of theſe Words. Common Senſe, therefore allows Time to partake of Quantity, as the Meaſure of the Continuance of Things in their Being. But perhaps you may aſk, whether Time was not before the World was created? And if Time does not flow in the Extramundane Space, where nothing is: A mere Vacuum? I anſwer, that ſince there was Space before the World was created, and that there now is an Extramundane, infinite Space, (where

God is preſent;) inaſmuch as there might have been of old, and now may be, ſuch, and ſo many Bodies, which then were not, and now are not; conſequently Time exiſted before the World began, and does exiſt together with the World in the Extramundane Space, becauſe 'tis poſſible that ſome Thing might have exiſted long before the World was made; and there may now be ſomething in the Extramundane Space, capable of ſuch a Continuance: Some *Sun* might have given Light long before; and at preſent this, or ſome other like it, may diffuſe Light thro' Imaginary Spaces. Time therefore does not imply an actual Exiſtence, but only the Capacity or Poſſibility of the Continuance of Exiſtence; juſt as Space expreſſes the Capacity of a Magnitude contain'd in it. But you may perhaps wonder why I explain Time without Motion, and will ſay, does not Time imply Motion? I anſwer no, as to its abſolute and intrinſic Nature; any more than it does Reſt. The Quantity of Time, in itſelf, depends not on either of them; for whether Things move on, or ſtand ſtill; whether we ſleep or wake, Time flows perpetually with an equal Tenor. If you ſuppoſe all the fixed Stars to have ſtood ſtill from their Beginning; not the leaſt Portion of Time wou'd be loſt by this; for ſo long as that Reſt continues, ſo long has this Motion flowed. There may be what we may call *firſt* and *laſt*, beginning and ending together, (with Regard to the firſt Appearance and Diſappearance of Things) even in that State of Tranquillity, which ſome Mind more perfect than ours may poſſibly comprehend. But as Magnitudes themſelves are abſolute *Quantums* Independent on all Kinds of Meaſure, tho' indeed we cannot tell what their Quantity is, unleſs we meaſure them; ſo Time is likewiſe a *Quantum* in itſelf, tho' in Order to find the Quantity of it, we are obliged to call in Motion to our Aſſiſtance, as a Meaſure whereby we may eſteem and compare the Quantity of it; and thus Time as meaſurable ſignifies Motion; for if all Things were to continue at Reſt, it would be impoſſible to find out by any Method whatſoever how much Time has elaps'd; and the ſeveral Ages wou'd roll on imperceptibly and undiſtinguiſh'd. Do I ſay we ſhou'd not perceive how Time flows? No indeed, nor any Thing elſe, but remain like Stocks or Stones in a continual Inſenſibility. We perceive nothing, unleſs ſo far as we may be inſtigated by ſome Change affecting the Senſes, or that our Souls are mov'd and excited by the internal Operation of the Mind. We eſteem the Quantities and different Degrees of Things according to the Extenſion or Intenſion of Motions ſtriking upon us either interiorly or exteriorly: So that the Quantity of Time ſo far as we can obſerve, depends upon the Extenſion of Motion. 'Tis not improperly obſerv'd by *Lucretius.*

Nec per ſe quenquam tempus ſentire fatendum eſt.
Semotum ab rerum motu placidâque quiete.

No Thought can think on Time; that's ſtill confeſs'd:
But thinks on Things in Motion or at Reſt.

Nor by the *Philoſopher himſelf* Ὅταν γὰρ αὐτοὶ μηδεν μεταβαλλωμεν τὴν διανοιαν, ἢ λάθομεν Phyſ. 16.4 μεταβάλλαντες, οὐ δοκεῖ ἡμῖν γεγονέναι χρόνος. *When we wake we cannot perceive or tell how much Time has paſſed during our Sleep*: which is certainly true: But it cannot be juſtly inferr'd from thence, Φανερὸν ὅτι οὐκ ἐστιν ἄνευ κινήσεως καὶ μεταβολῆς ὁ χρόνος. *We do not perceive the Thing, therefore there is no ſuch Thing*, that is a falſe Illuſion, a deceitful Dream, that wou'd cauſe us to join together two remote Inſtants of Time. But nevertheleſs this is very True; ὅση ἡ κίνησις τοσοῦτος, καὶ ὁ χρόνος ἀεὶ δοκεῖ γεγονέναι. That is, *for as much Motion as there was, ſo much Time ſeems to have been elapſed*; nor, when we mention ſuch a Quantity of Time, do we merely mean any Thing elſe, than the Performance of ſo much Motion, to the continued ſucceſſive Extenſion of which we imagine the Permanency of Things is coextended. Moreover, becauſe we conceive Time to flow always in an equal Channel, not by Starts, ſome times ſlower, and faſter at others. (For were ſuch Diſparity allow'd, it wou'd admit of no Manner of Computation or Dimenſion.) Every Motion muſt therefore not be conſider'd as equally fit to determine and ſhew the Quantity of Time, but that chiefly which being the moſt ſimple and uniform, goes on always with an equal Pace; the Thing moved every where retaining an equal Force, and being carried along in a uniform Medium. Such a moveable Body muſt therefore be pitched upon for the Determination of Time, as at leaſt conſtantly keeps an equal Impetus, with Regard to the Periods of its Motion, and runs thro' an equal Space: and indeed, for common Uſe, the moſt remarkable Motion poſſible ought to be taken; ſuch a one as is immediately obvious to all, and ſtrikes the Senſes of every one; of this Kind is the Motion of the Stars, and particularly of the *Sun* and *Moon*, which obſerve a wonderful Regularity one to the other, in all Things, and are conſpicuous to every Part of the Earth; thoſe Bodies being deputed for that Purpoſe, not only by the common Conſent of Mankind, but adapted to it by the divine Will of the Creator, who pronounc'd as follows: *Let there be Lights in the Firmament of the Heaven, to divide* Gen. 1. 14.

the Day from the Night: and let them be for Signs, and for Seaſons, and for Days, and for Years. But you may aſk, how we *can know that the Sun moves with an equal Motion, that the Time of one Day or Year, for Suppoſition Sake, is exactly equal to that of another?* I anſwer, we cannot know this any otherwiſe (except what we gather from the divine Teſtimony) but by comparing the Motion of the Sun with other equal Motions. If, for Example, the Sun's Motion on a Sun-Dial (which almoſt to a Certainty, and in an exquiſite Manner, ſhews the Quantities of the Spaces run through by the Sun in Circles parallel to the Equator) be found to agree with the Motions of any artificial Time-Keeper accurately made, the Sun's Motion muſt conſequently be equal. For ſince the very Conſtruction of the Machine is ſuch, as to equally move in every ſucceſſive Repetition of its Motion, as, for Inſtance, an Hour-Glaſs, deſtined to meaſure an Hour; and becauſe the Water or Sand contain'd in it remain entirely the ſame as to Quantity, Figure, and Force of deſcending, and the Veſſel that contains them, as likewiſe the little Hole they run thro' don't undergo any Kind of Mutation, at leaſt in a ſhort Space of Time, and the State of the Air much the ſame; there is no Manner of Reaſon for us not to allow the Times of every running out of the Water or Sand to be equal; If therefore the ſolar Motions, either as to whole Periods, or to the proportional Parts of them, are found to correſpond entirely with the repeated Motions of ſuch an Inſtrument, we thence may very juſtly conclude them to be entirely equal and uniform. Whence it ſeems to follow (which poſſibly wou'd appear wonderful to ſome,) that the celeſtial Bodies are not eſſentially, and properly ſpeaking, the primary and original Meaſures of Time; but rather thoſe Motions which are near us, that ſtrike upon our Senſes, and fall under our Experience, ſince by their Means, we diſcover the Regularity of the celeſtial Motions. Nor is even the Sun itſelf a proper Judge or Witneſs in this Affair, any farther than as its Veracity is ſhown, by the Atteſtation of an Horary Machine. Neither indeed can we know by any Methods, whether the Periods of the Stars, many Ages ſince, were altogether equal to their Revolutions in our Age; no one, for Inſtance, can pretend to aſſert as a certainty, that the Age of *Methuſalem* himſelf, who lived a thouſand Years wanting one, was really longer than that of a Man, who now dies before he arrives at an Hundred. Why might not the Sun, being then younger and more vigorous, have performed his Periods ten Times ſooner than at this Time? Perhaps the Air was then purer, and by that Means the Gravity of Bodies becoming more powerful, organical Inſtruments themſelves might have had ſwifter Motions, and ſo, if compared with the Inſtruments of our Time, would very much diſagree. *Empedocles* indeed, as we find in *Plutarch*, is ſaid to have been of Opinion, that the Sun in his Infancy cauſed the Days to be much longer than in his Time. But this does not ſeem agreeable to Reaſon, becauſe ſuch circular Motions are won't to Decay continually, rather than Increaſe. But to return from this ſcarce ſerious Digreſſion; the Quantity of Time, (or Continuance of Things in their Being, State, or Motion) is found out, as was before obſerv'd, firſt by any well known equal Motion, (or as to the Parts uſed for this Purpoſe, conſtantly equal and ſimilar to themſelves) and afterwards by any other Motions which upon Trial, proportionably anſwer to it, and chiefly by the celeſtial Motions, eſpecially of the Sun and Moon: So that thoſe Times are equal, when the ſame Hour-Glaſs is run out once, twice, or any equal Number of Times, or, in which the ſame Stars perform the ſame Period, or equal Parts of the ſame Period; but thoſe unequal, according to any Proportion, in which unequal Periods are ſimilarly or proportionably compleated. Nor ſhou'd any one object to Time's being commonly taken for the Meaſure of Motion and conſequently in bringing to this the Differences

of Motion, ſwifter, ſlower, accelerated, or retarded. Time muſt be taken as foreknown; [yet the Quantity of Time is not therefore to be determined by Motion, but the Quantity of Motion by Time;] for there is no Reaſon why Time and Motion may not mutually perform theſe Offices. For, in like Manner as we firſt of all meaſure a Space by ſome Magnitude, and declare it is ſo much; and afterwards by Means of this Space, compute other Magnitudes correſpondent with it: So we firſt aſſume Time from ſome Motion, and afterwards judge thence of other Motions, which in Reality is no more than comparing ſome Motions with others, by the Aſſiſtance of Time; juſt as we inveſtigate the Ratios of Magnitudes by the help of ſome Space. For Example, he who computes the Proportion of Motion, by the Proportion of Time, does no more than get the ſaid Ratio of Motions from Clocks, Dials; or from the Proportion of ſolar Motions performed in the ſame Time. This *Ariſtotle* doubtleſs knew, and has plainly taught: ȣ̓ μόνον, (inquit) τὴν κινησιν τῷ Phyſ. 18 4.
χρονῳ μετρȣ̃μεν, ἀλλὰ ϗ τῇ κινήσει τ͂ χρόνον διά τὸ ὁρίζεσθαι ὑπ᾽ ἀλλήλων. That is, *We not only meaſure Motion by Time, but alſo Time by Motion, becauſe they determine each other.* Again, becauſe Time, as has been ſhewn, is a Quantum uniformly extended, all whoſe Parts correſpond, either Proportionably to the

the reſpective Parts of an equal Motion, or to the Parts of Spaces moved through with an equal Motion; it may therefore be very aptly repreſented to our Minds, by any Magnitude alike in all its Parts; and eſpecially by the moſt ſimple ones, ſuch as a ſtrait or circular Line; between which and Time there happens to be much Likeneſs and Analogy. For as Time conſiſts of Parts altogether ſimilar, it is reaſonable to conſider it as a Quantity endowed with one Dimenſion only; for we imagine it to be made up, as it were, either of the ſimple Addition of riſing Moments, or of the continual Flux of one Moment, and for that Reaſon aſcribe only Length to it, and determine it's Quantity by the length of the Line paſſed over: As a Line, I ſay, is looked upon to be the Trace of a Point moving forward, being in ſome ſort diviſible by a Point, and may be divided by Motion one Way, *viz.* as to Length; ſo Time may be conceiv'd as the Trace of a Moment continually flowing, having ſome Kind of Diviſibility from an Inſtant, and from a ſucceſſive Flux, inaſmuch as it can be divided ſome how or other. And like as the Quantity of a Line conſiſts of but one Length following the Motion; ſo the Quantity of Time purſues but one Succeſſion ſtretched out as it were in Length, which the Length of the Space moved over ſhews and determines. We therefore ſhall always expreſs Time by a right Line; firſt, indeed, taken or laid down at Pleaſure, but whoſe Parts will exactly anſwer to the proportionable Parts of Time, as its Points do to the reſpective Inſtants of Time, and will aptly ſerve to repreſent them. Thus much for Time. We now proceed to the effective Force of it, which doubtleſs may juſtly be conſider'd as a kind of Quantity, capable of Computation like other Quantities (whatever is it's Origin and Nature, for theſe we leave to *Philoſophers.)* For it is very plain from Experience, that when two moveable Bodies depart from the ſame Place along the ſame Line, the one often overruns the other, or moves a greater Space, in the ſame Time; the Reaſon of which can be only this, that that Body which moves ſwifteſt, is acted upon by a greater Force, or motive Power. And as it is manifeſt, that there can be no Reaſon for us to object why this Exceſs of Spaces moved through together, may not happen in all Kinds of Proportions; this Force may therefore be conceiv'd juſtly as divided into any Parts (in the ſame Manner as the Intenſive Parts of any Kind of Quality may be, and are, for Diſtinction's ſake, uſually call'd Degrees) it may, I ſay, be conceiv'd as diviſible into any infinite or indefinite Parts; the common Boundary which joins them to each other, and diſtinguiſhs them, or the Part abſolutely the leaſt, (upon the

Suppoſition, that, Quantities conſiſt of infinite Atoms) being call'd Reſt, or the loweſt Degree of Velocity; from the Increaſe or continual Flowing of which any Degree of Velocity may be ſuppoſed to be generated, exactly in the ſame Manner as we imagine a Line to be produced by the Appoſition of Points or by Motion; or Time, by the Succeſſion or Flux of Inſtants. Conſidering therefore the Thing abſolutely, in order to repreſent the Quantity of this Force juſtly to the Mind, we need only lay down but ſome regular Magnitude in its ſtead, (ſuch a one in whoſe Parts we can eaſily and immediately diſcover any Difference and Proportion.) As a right Line is the moſt Simple and Perſpicuous of any, 'tis therefore the fitteſt to repreſent any Degree thereof. Thus indeed this Force, generally and abſolutely conſider'd in it ſelf, does not imply Time, and can be conceiv'd without it (for in any Inſtant or interval of Time, a moveable Body may be ſuppoſed to be endeued with it) unleſs ſo far as it is computable, and comes under a Mathematical Conſideration, in which Senſe it is called Velocity, when Time ſignifies Space; whence, for Example, its Quantity is diſcover'd and determin'd. Therefore Velocity is defined to be that Power, by which a moveable Body can paſs over a given Space in a given Time; whence it follows, that every particular Quantity of any Velocity, cannot be known, neither by the Space moved through only, nor, by the Time ſingly, (for any Velocity may trace out a given Space in a given Time) but may be found by Calculation from the Quantity of Space and Time together; as on the contrary, the Quantity of Time may be obtain'd from the Quantity of the Space and Velocity together: Nor does the Quantity of Space (ſo far as it can be known this Way by Motion,) depend wholly upon the Quantity of a definite Velocity, or upon any aſſigned Time, but upon the Conjoint-Ratio of both. And indeed, that we may explain further, what Regard theſe have to each other, the Quantity of Space (ſo far as can be this Way computed,) is found much after the ſame Manner, as we do that of a Superficies by its Dimenſions, *viz.* by the Quantity of one Line (expreſſing its Length or Height) and by the Quantities of every parallel right Line, which paſſing thro' each Point of that right Line, do in a Manner conſtitute and make up the whole Superficies, or at leaſt limit and determine it; that is, as it were by the drawing of each of them into the reſpective Points of the ſaid Line. But the Quantities of Velocity and Time, are found exactly after the ſame Manner, as when a Superficies, and one of its Dimenſions being given, we thereby find the other Dimenſion. I ſay, how much it is whereſoever you pleaſe, or at leaſt ſome where, for it may happen, that the other Dimenſion, ſo far as it is ſpread thro' all the Points of the former Dimenſion, will be diſſimilar, and deformed in every Part thereof; which I would have appear by the Tract, for this uſual Conſideration requires a clear Explanation. To every Inſtant, or indefinitly ſmall Particle of Time; I ſay, to every Inſtant, or indefinitly ſmall Particle, for as it not in the leaſt ſignifies, whether a Line be conceived to be made up of innumerable Points, or of indefinitly ſmall Lines, ſo it is the ſame Thing, whether Time be ſuppoſed to conſiſt of Inſtants, or of innumerable ſmall Times. We at leaſt for Brevities ſake, make no Scruple to take Inſtants, as ſo many very ſmall Times, that is, Points for very ſmall Lines, repreſenting exceeding little Times. I ſay, to every Moment of Time there anſwers ſome Degree of Velocity, which a moveable Body is then to be conceived to have; to which Degree, ſome Length of the Space moved over anſwers. (For here the moveable Body is conceived only as a Point, and ſo the Space moved thro' can be but a Line.) But, becauſe Moments of Time can really have no Dependence upon each other, we may ſuppoſe a moveable Body in the next Inſtant, to have ſome other Degree of Velocity, (either equal to the former, or different from it in any Proportion,) to which accordingly will anſwer ſome other Length of the Space moved through, reſpecting the former in ſuch Proportion, as this Degree of Velocity does that preceding it. For ſince Inſtants of Time are entirely equal to one another, the Ratio of the Lengths of the Spaces, will depend wholly upon the Ratio of the Velocities, and accordingly will be equal and ſimilar to it, (becauſe, if this were not true, Velocity could in no wiſe be meaſur'd; ſince from the ſole Proportion of the Spaces moved through in the ſame Time, (or in the ſame Inſtant,) the Velocity is immediately or mediately had, and the one ſaid to be ſo much in Reſpect to the other.) In like Manner, if in all the Moments of any given Time, be aſſigned Degrees of Velocity anſwering to them, the Aggregate of theſe, will form a Quantity, any of whoſe Parts are exactly proportionable to the reſpective Parts of the Spaces moved through, that is, the ſmall correſpondent Parts in the ſame Times, and ſo the Magnitude repreſenting the Quantity of theſe Degrees may alſo repreſent the Space moved through; for as much as it exactly expreſſes, what Proportion the Parts of the Space moved through in each particular Time, do obſerve among themſelves, and therefore when it flows equally, will be moſt aptly repreſented by a right Line, as we have already ſaid, and the ſeveral Degrees of Velocity, in the Inſtants of Time, whether equal or unequal in each Inſtant, may be alſo expreſſed by right Lines, and, becauſe theſe Degrees of Velocity do in every Moment of Time paſs over one another independently, and without Mixture; therefore, if right Lines be drawn through all the Points of the Line, repreſenting the Time, in ſuch Manner as not to meet, that is, all Parallel to each other; the plain Superfices thence reſulting, will exactly repreſent the Aggregate of the Degrees of Velocity, (according to the Quantity of Time, and the determinate Ratio of the ſuppoſed Degrees of Velocity,) which Superficies having its Parts proportionable to the reſpective Parts of the Space moved through, may very well repreſent the ſame. And this Superficies for Brevity's ſake, I call henceforwards *aggregatical Velocity*, or the *Repreſentative* of Space. Here, by the way, I would not, that any Body ſhould be concern'd, that I have ſaid ſome Length is moved through in every Inſtant of Time, and ſo affirm, as it were, that there is ſuch a Thing as inſtantaneous Motion. For if is Time be ſuppoſed to be made up of Moments, Lines will be compounded of Points too; and if unequal

Lines be made up of infinite Points, equal in Number, it muſt follow neceſſarily, that the Points of Lines are unequal, in the ſame Proportion to the Lines, and ſo unequal Points of this Kind, of which the whole Length mov'd thro', is as it were made up, are to be conceived only as Lengths moved thro' in equal Moments of Time. But if this be not an agreeable Way of ſpeaking, and inſtantaneous Motion, will not at all be allowed, know ye, that by Inſtants, I mean no other than infinitely ſmall Parts of Time; each having ſome Degree of Velocity correſponding therewith, to which an infinitely ſmall Part of the Space moved through is proportionable; and then to repreſent any one of theſe particular Degrees, continuing the ſame for a very ſmall Time, we muſt ſubſtitute a little Rectangle apply'd to the ſaid ſmall Time, inſtead of a right Line. And as the ſame Conſequence will follow from either of theſe Conſiderations, and the firſt ſeems to be the moſt plain and evident, it is this therefore that we ſhall hereafter inſiſt upon. But to return. Becauſe parallel right Lines may be drawn through all the Points of a right Line, either equal or unequal as you pleaſe, forming a plane Superficies, therefore we may apply ſeveral different Degrees of Velocity, to every Inſtant of Time, either equal or unequal, according as the moveable Body, throughout its whole Courſe, is ſuppoſed to have the ſame, or various Impetus's, any how increaſing or decreaſing. If it is ſuppoſed to have the ſame Velocity always, it eaſily appears from what has been ſaid, that the aggregatical Velocity in any given Time, may very aptly be repreſented by a Parallelogram, as AZZE, the Side AE expreſſing the given Time, and the other Side AZ with its Parallels BZ, CZ, DZ, EZ, will ſhew every Degree of Velocity in each Moment of Time, being all in this Caſe equal. Alſo the Parallelograms AZZB, AZZC, AZZD, AZZE, will repreſent the Spaces moved through in the reſpective Times AB, AC, AD, AE. From which Conſideration only, may immediately be deduced all the Properties of an equable or uniform Motion, as they call it; as for Example, that *the Spaces moved through with an equable Velocity, are to one another as the Times*: That *the Spaces run through in the ſame Time, are to one another as the Velocities*; and contrarywiſe: If the *Spaces are as the Velocities, the Time will be equal, if as the Times, the Velocities will be equal.* And *if the Spaces are equal, the Times will be reciprocally as the Velocities*; and on the contrary, *if the Times be reciprocally proportionable to the Velocities, the Spaces will be equal to one another.* And

Fig 1.

laſtly, *any Spaces are to one another in the Ratio, compounded of thoſe of the Velocities and Times*; alſo, *if the Ratio of the Times be taken away from the Ratio of the Spaces, the remainder will be the Ratio of the Velocities; or if the Ratio of the Velocities be taken away, there will remain the Ratio of the Times.* For theſe are the Affections of Parallelograms compared to one another, (that is, of equiangular Parallelograms; for when theſe repreſentative Parallelograms are compared to one another, they muſt be equiangular, tho' indeed, when ſeparately conſider'd, it matters not what Angle is ſuppoſed) I ſay, theſe are evident from the Nature of Parallelograms, and ſo eaſily flow from what has been ſaid, as to ſeem to require no other Demonſtration. And indeed, as to all Matters falling under a Mathematical Conſideration, (that is, any how involving Quantity,) Theorems do appear with great Facility, and may be moſt compendiouſly demonſtrated, if any one knows rightly, and congruouſly, how to reduce Quantities of any Kind ſoever, ſubject to his Contemplation, to analogous Magnitudes. But again, if the Degree of Velocity be ſuppoſed to be perpetually augmented, or diminiſhed throughout all the Inſtants of Time, from the leaſt Degree, *viz.* Reſt, to a given Degree of Velocity, or from this given Degree to Reſt. The aggregatical Velocity may in like Manner be expreſſed by any Superficies, perpetually increaſing from a Point to a Line of a given Magnitude; or decreaſing back again, from Fig. 1. this Line to that Point; but beſt of all by a right Lin'd Triangle, ſuppoſe the Triangle AEY, wherein the Side AE denotes the Time, and the parallel Lines BY, CY, DY, EY applied to the Points thereof, the ſeveral Degrees of Velocity, anſwering to every Inſtant of Time, equally increaſing from the Point A, (repreſenting Reſt, or the leaſt Degree of Velocity,) to a given Degree repreſented by the greateſt Line EY, or decreaſing back again from the ſaid Line EY, to the Point A, repreſenting Reſt. And for the ſame Reaſon as before, the Triangles ABY, ACY, ADY, AEY will repreſent the Spaces moved through, from the Beginning of the Motion, in the reſpective Times. And conſequently, if the Velocity be ſuppoſed to be equally augmented or diminiſhed from any one given Degree to another, both the aggregatical Velocity, and anſwerable Space moved through, may be repreſented by the Trapezium CYYE. Hence all the chief Properties of this Kind of Motion, (moſt aptly call'd by *Gallilæo*, a Motion uniformly accelerate,) may very eaſily be found out and demonſtrated. Such as, that *thoſe Spaces are equal, which are mov'd through in the ſame Time, by a Motion uniformly accelerate from Reſt, and a uniform equable Motion, provided the Velocity of the latter be one half the greateſt Velocity of the former. That the Spaces moved through by a Motion uniformly accelerated from Reſt, are to one another, as the Squares of the Times, (or in the Duplicate Ratio of the Times.)* And in comparing after this Manner, different accelerate Motions: *That the Spaces moved through, are in a Ratio compounded of the Ratio's of the Times, and of the greateſt Velocities:* And ſuch like as theſe, either connected with them, or following from them, which agree to Triangles compared together, as to their Ratio's, and that of their Sides; which may eaſily be perceived and demonſtrated, from what has been laid down. Moreover, if the Degrees of Velocity, in a continual Succeſſiſion from Reſt, throughout every Inſtant of Time, to a given Degree, be conceived to increaſe,or to decreaſe backwards from that, in the Progreſſion of the ſquare Numbers, the aggregatical Velocity, as well as the Space moved through, may moſt conveniently be repreſented by the Complement of a *Semi-parabola*, as AEX, whoſe Vertex A is ſuppoſed to be Reſt, (or the Beginning of the Motion and Time,) the Abſciſs

Fig. 2.

Abſciſs AE a given Time, and Line BX the firſt Degree of the increaſing Velocity (being as 1.) The next CX the ſecond Degree (being as 4) The next DX (being as 9.) And ſo on to the laſt EX: Which is manifeſt from a well known Property of the Parabola. In like Manner, any ſuppoſed Degrees of Velocity, any how increaſing or decreaſing continually, or interruptedly, I ſay, after any imaginable Way, may be truly and conveniently expreſſed by right Lines applied to that repreſenting the Time, keeping whatever Proportion any one is pleaſed to aſſign; ſo that knowing from thence, the Meaſure of the repreſentative Space, the Quantity of the Space moved through, will be eaſily had; and contrarywiſe, if the Nature of the ſaid Space be ſhewn, you may ſome how or other find the Velocity and Time; which Things certainly neceſſary to be underſtood hereafter by Learners, and appearing of great Uſe throughout the whole Theory of Motion, we have thought fit to handle ſomewhat largely. Which Buſineſs being diſpatched, I advance no farther at this Time.

LECTURE II.

HAVING attempted to conſider the different Ways by which Magnitudes are conceived to be produced; we have begun by touching upon that firſt and principal Way, which is performed by local Motion. But ſince it is common with *Mathematicians*, to aſſume, as poſſible, Motions after various Manners, from which Magnitudes of ſeveral Sorts do reſult, we make Choice of two Springs, from whence thoſe Differences of Motions do flow, *viz.* the *Mode* of *Lation* (or *Baring*) itſelf, and the Quantity of the *Motive Force*; the latter of which not being ſo obvious and apparent, we have lately endeavour'd to explain and ſet in a clear Light. We ſhall now ſpeak further of their ſeveral differences, and ſhew how they may be ſerviceable to the Generations of Magnitudes. In regard to the *Mode* of *Lation*, Magnitudes are generated, either by ſimple Motions, or by compound Motions, or elſe by the Concurrence of Motions (for I make a Difference between Compoſition, and Concurrence, tho' theſe by ſome are taken for the ſame.) And firſt of all, let us ſee what are the *Hypotheſes*, and effects ariſing from ſimple Motions. Of theſe there are two Kinds, φορὰ, and περιφορὰ, *Progreſſion* and *Circumlation* or *Circumrotation*. Under the former of which is comprehended all Motion, not regarding any fixed Place, (by the Name of Place, I underſtand any Magnitude, reckoning even a Point as ſuch) to which it is, as it were, tied and confined; whether that Motion be direct, reflected or refracted; or whether it keeps on in a beaten Path; inconſtantly leaps, or irregularly wanders. But becauſe Art cannot in the leaſt take Cognizance of any Thing which is irregular, it is ſufficient for *Mathematicians* to conceive, that any Magnitude may move on according to any aſſigned Direction or Path. As for Example, a Point along a *right Line*, *circular Line*, *elliptick Line*, a *Spiral*, &c. But the chief *Hypotheſes* firſt fram'd by *Mathematicians* about theſe kind of Motions, being of the greateſt and moſt frequent Uſe to the Formation of Magnitudes, are theſe: That a Point may move directly on, as longs as you pleaſe, from an aſſign'd Term, in a right Line, by which Motion it is evident a right Line is deſcribed. That a right Line may ſo move along any other Line, as in the mean while, to perpetually keep a parallel Situation, (*viz.* that the ſame according to the Poſition which it happens to have in any Moment of Time, may be parallel to itſelf, when ſituate in any other Moment of him.) Alſo, that any Line (definitely or indefinitely extended, which we would have underſtood in all) may proceed forwards, with a direct Motion, likewiſe parallel to itſelf, (I ſay, by a direct Motion, that is, that every Point thereof may deſcribe right Lines,) which two Motions are doubtleſs equivalent to each other, and produce the ſame Effect, and thoſe Superficies conceiv'd to be generated by either of theſe, which are the moſt equable and uniform, ought to have the Preference of the reſt; ſuch as *Parallelograms*, in a Plain, (either right lin'd, or mixed) and *priſmatick* and *cylindrical Superficies*, taken both it a ſtrict and large Senſe, delineated in a Solid, if I may ſo ſpeak, or not in one Plain. Let, for Example, in the firſt Place, BC be a right Line, and let the right Line AB ſtanding upon it, move along perpetually parallel Fig. 3.

to itſelf, until the Point B being arrived at C, the right Line AB coincides with DC parallel to AB. It is manifeſt by this Motion, a plain Figure being the *Parallelogram* ABCD will be *generated.* It is alſo evident, that any aſſum'd Point in AB, ſuppoſe E, does deſcribe a right Line, the Parts EE of which intercepted by the right Line AB, are equal to the Parts BB of the right Line BC, intercepted by the ſaid right Line AB reſpectively, (that is moved through by the Point B in the Fig. 4. ſame Time.) Nor is it leſs manifeſt, if on the contrary, the right Line BC moves along BA, the ſame Superficies will be deſcribed; and all Points, as F, of the right Line BC deſcribe right Lines; as likewiſe the Parts FF of theſe intercepted by the Parallels BC, are equal to the reſpective Parts BB of the Line AB. (But, for Brevities ſake, I would have it henceforwards obſerved, as well in theſe as the like Caſes, I call that one of theſe Lines, by whoſe Motion a Magnitude is deſcribed, *generative*; and the other according to which the former moves, or on which it ſtands, I give the Name of *directive*, becauſe the Courſe of the Line moved, is governed by Fig. 5. it, or accommodated to it.) Again, let there be any Curve, (as the Arch of a Circle) BC, and the right Line AB ſtand on the ſame, in the ſame Plain; and let the right Line AB continually move along the Curve BC parallel to itſelf, until the Point B arrives at C, and the right Line AB at length coincides with DC, parallel to AB in its firſt Situation; by this Motion alſo will be deſcribed a plain *Figure* being a *Parallelogram*, (in a large Sence.) Becauſe the oppoſite Sides thereof are parallel; the right Line AB, to the right Line DC; and the Curve AD, to the Curve BC. For here too, every Point of the *generative right Line*, as E, will deſcribe Lines ſimilar and equal to the *directive Line* BC; as well the Wholes, as the Parts intercepted by the ſame Parallels AB. For if any two Points EE, be join'd by a right Line, and the Points BB anſwering to them, be alſo join'd by a right Line; becauſe the right Lines EB are equal to each other, (for they are no other but the very ſame Line in a different Situation,) and parallel by Suppoſition, the right Lines EE, BB ſhall be equal and parallel, therefore it is evident, the Curves EE, BB are equal and ſimilar to each other; they are equal, becauſe all the Subtenſes EE are ſeparately equal to the Subtenſes BB, and ſimilar, becauſe the right Lines AB do conſtitute equal Angles, with the adjacent reſpective Subtenſes EE and BB, and ſo the right Lines EE equal to thoſe which the right Lines BB are, thoſe by themſelves, and theſe Fig. 5.

by themſelves (for the Similitude of Lines, and any Magnitudes whatſoever, conſiſts in the Proportionality of ſuch Parts, and Equality of ſuch Angles, as perhaps we have ſhewn elſewhere more
Fig. 6. clearly and fully.) But if on the contrary, the Curve Line BC, he made the *generative* Line, and the right Line BA the *directive* Line, that is, if BC be moved along, BA, parallel to itſelf, there will be produc'd the ſelf ſame *parallelogramick Superficies*; and every Point as F of the right Line BC, will deſcribe right Lines parallel to BA; as alſo the Parts FF, will be equal to the reſpective Parts BB; as follows after the ſame Manner, from the Suppoſition of the conſtant Paralleliſm of the Curve BC. Laſtly, let there be any Curve, (either made up of right Lines, making right Angles, which alſo deſervedly may bear the Name of a Curve; at leaſt *Archimedes*, calls Lines compounded of right Lines as the Perimeters of Figures inſcribing or circumſcribing Circles, καμπυλῶν γραμμῶν, inflected Lines; as on the contrary, any Curve Line may be looked upon, as made up of innumerable ſuch Lines, indefinitely ſmall, being adjacent, and making Angles one after another.) I ſay, let there be ſome ſuch Curve BC, conſtituted in one Plain, then in another Plain, elevated any how above the Plain of the

Line

along Parallel to it ſelf, in the Manner above ſhewn; by this Motion a *cylindrical Superficies* will be deſcribed (or certainly a *priſmatical Superficies*, if the *directive* Line be ſuppoſed to be made up of right Lines:) A *cylindrical* Superficies, ſtrictly ſpeaking, when the *Directive* Line is *Circular* or *Elliptick*; but ſuch in a large Senſe, when the Line is of another kind, as *Parabolical*, *Hyperbolical*, &c. In this Motion too every Point of the *generative Line* will deſcribe Lines ſimilar and equal to the *directive Curve*; equal, becauſe E B are equal and parallel; and ſo E E, B B equal and parallel too; but ſimilar, becauſe the Angles E E E, B B B are equal. So likewiſe on the contrary, the ſame Superficies will be deſcribed by ſuppoſing the Curve BC to move along the right Line A B parallel-wiſe. Alſo how all the Points of the Curve B C will delineate right Lines parallel and equal to the reſpective intercepted Parts of the right Line A B, may be ſhewn after the ſame Manner as before in the Example of a plane Figure. Therefore if the Superficies thus generated be cut by any Plane parallel to a Right Line, *Directive* or *Generative* (which in every Situation may be call'd a ſide of the Superficies generated) the common Section will conſiſt of two parallel Lines equal to one ano-

10, 11. *Elem.*

ther. But as to the Superficies thus generated, it is worth Obſervation (for our Deſign is not only to treat of the bare Generations of Magnitudes; but likewiſe ſhew ſome of their general Properties reſulting from the different Ways of Generation) if the generative Line be ſtraight (as diſtinguiſhed in the Figure by the Letter Z) the Parts of the Superficies produced lying between Lines parallel to the generative Line, are always proportionable to the reſpective Parts of the directive Line, *viz.* (the Superficies BCCB to the reſpective right Lines BB.) But if the directive Line be a Curve (as Y in the Figure) it will not always happen, that the Superficies intercepted by the *generative right Line* are proportional to the Parts of the Curve intercepted by the directive Line; but this happens at leaſt when the generative right Line is every where inclin'd, equally in every Point thereof, to the Curve BC; as it falls out in the Superficies of every right Cylinder, Laxly or Strictly ſo called. Becauſe then the generative right Line in all Points of the Curve is perpendicular to the Tangents or Subtenſes drawn to thoſe Points. But Fig. 6. if, for Example, the Curve BC be ſuppoſed the Arch of a Circle, which is equally divided in the Point B, the Superficies ABBA ſtanding upon the equal Peripheries BB will not neceſſarily be equal to one another, becauſe (in all Caſes except thoſe before-mention'd of the right Cylinder) the right Lines AB are every where unequally inclined at the Points B, (in comparing any one with another;) that is, they make unequal Angles with the ſeveral Tangents at B, and Subtenſes BB. On which thing depends *that inſuperable Difficulty* wherewith thoſe have been troubled, *who have endeavoured to meaſure oblique cylindrical Superficies*, or *compare them as to proportion with the Superficies of right Cylinders, or other Superficies in ſome ſort known*. *Laſtly*, We ſuppoſe any plain Superficies in like manner to move ſtraight onwards parallel to itſelf, *viz.* ſo that every Point of it does deſcribe right Lines equal and parallel to one another; or that all right Lines thereof do mark out *plain* parallelogramick Superficies (which is what follows from hence;) by this Motion will be deſcribed priſmatical or cylindrical Solids; the very ſelf ſame, concerning the Superficies whereof we have but now handled, and to which by a like Reaſoning may be apply'd what we have ſhewn agrees to thoſe Superficies. As, that the Superficies of them intercepted by parallel Planes and the Bodies themſelves contained under them, and their ſides, (or the reſpective

Parts of the directive right Line) are proportionable. Alſo if finite Bodies of this Kind be cut by Planes parallel to any one of the ſides, the common Sections will be *Parallelograms* (ſuch as EEBB.) But that we may mention many Things in a few Words, *whatever has been any where demonſtrated concerning the Superficies and Solids of Priſms and Cylinders, ſtrictly taken as ſuch*, does for the moſt Part, by obſerving a juſt Analogy, agree to all Quantities generated after this manner, nor does any thing elſe about progreſſive Motion offer it ſelf at preſent to be added; for I deſignedly am ſilent concerning ſome Things of difficult Explanation. Again, the other Kind of ſimple Motion, made uſe of in *Mathematicks*, is *Circumlation*, or *Circumrotation*, which for Example is then made, when ſomething of the Magnitude moved, (as ſuppoſe any Point of a Line, or Line of a Superficies) remains fixed and immoveable, while the whole Magnitude remaining as it were tied and bound to the ſame, is carried about according to any aſſigned Direction. The moſt general Property of which Motion is, that all Points of the moveable Magnitude while they move tranverſely in any one Plane, do every one deſcribe the Peripheries of Circles; and indeed all moving in one and the ſame Plane paſſing through a fixed Point, are Parallel, and Concentrick, and ſimilar to each other; but thoſe in different Planes are ſimilar, or not ſo, according as the arbitrary Diverſity of *Hypotheſes* requires. But that Circumlation is the moſt convenient and natural, where every Point of the Magnitude mov'd will deſcribe circular Peripheries of one and the ſame Circle; that is, when they are all carried round in the ſame Plane; being doubtleſs ſuch as Nature it ſelf conceives and follows, whereby Magnitudes are kept to their ſeveral immoveable Holds, and hindred from flying off in right Lines which they naturally would do; as appears in the Motion of Pendulums and Bodies in Slings; or even when any Object being reſiſted cannot eaſily keep in a ſtrait Path; as appears to be the Caſe of the Motions of *Wheels*, *Whirlpools*, *Whirlwinds*, *and perhaps of the Stars themſelves.* But the general Nature of theſe and ſuch like Motions, cannot be ſo ſoon explained in Words; therefore, it is neceſſary to relate the principal Hypotheſes they afford. And firſt of all, it is granted a right Line, in a Plane can be moved about any Point fixed in it; by which Motion it is evident all Points of the moving Line deſcribe the Peripheries of Circles; all parallel and ſimilar to one another. As if the right Line AB be moved about the

Fig 7. fixed Point C, each of the Points A,E,B, will deſcribe circular Peripheries AA,EE, BB, parallel to themſelves and all ſimilar, (*viz.* ſubtended under the ſame or equal Angles, whereof C is the common Center or Vertex) And it is manifeſt that after this Way will be generated Circles, and Sectors of Circles (as ACA,BCB) and annular Planes too, one of which is *that* remaining after the leſſer concentrick Circle EEEE is taken from the greater AA BB. From which Generation we infer that the Area's of Circles and circular Sectors are made up of ſimilar and concentrick circular Peripheries, as many in Number as there are Points in the Radius; by means of which may be eaſily deduced the well-known Menſuration of the Area of a Circle. 2*dly*, It is ſuppoſed that any right Line, indefinitely extended, having one Point thereof fixed, may Fig. 8. revolve about any given Line, either a Curve or conſiſting of right Lines conſtituted in ſome other Plane, ſo as to always touch this Line, or as it were ſtick to it. For Example, let AB be a right Line indefinitely extended, and V a fixed Point in the ſame, and let the right Line AB paſſing always thro' V move along any Line BC in another Plane; *viz.* ſo that ſome Point of the moveable Line continually adheres to the Line BC; by this Motion

tion will a curve Superficies be produced (at leaſt conſiſting of Planes, which in a general Way we make no Scruple to call Curves, ſince *Archimedes* has done it before) which if the whole directive Line conſiſts of right Lines of a finite Length, there will be formed a *pyramidal Superficies*, being a Number of Triangles meeting in the Vertex V; but if it be circular, or ſome one of the conick Sections, the Superficies will become ſtrictly *Conical*; but if any other Line, it may at leaſt be ſaid to be *Conical* in a large Senſe; and by ſome is ſo called. A Property of which Superficies evidently following from its Generation, is, that if it be cut by a Plane thro' the fix'd Point V, the common Section of it and the Superficies, will be a right-lined Angle. For if the Plane cutting the ſame thro' V, meets the directive Line in two Points, as D, E (and it will meet the ſame in two Points, otherwiſe it will not cut that Superficies) the right Lines VD, VE being drawn will be both in the cutting Plane and the Curve Superficies: In the Plane, from the Nature of a Plane; in the Superficies, becauſe the ſame generative right Line paſſes thro' the Bounds of theſe, and ſo coincides with them. But in a Motion of this kind, ſuppoſing right Lines drawn from the fix'd Point V (or Vertex) to the di-

Fig. 8.

rective Line to be unequal, it is enough manifeſt the Line BC will not be delineated or moved over by the Point B, becauſe unequal Lines (as VB, VE, VC) cannot agree between themſelves; and ſo the Point B moving on will fall above or below the Points B,E,C; neither, ſuppoſing this inequality, will any other Point in VB how far ſoever extended, (ſuppoſe G,) deſcribe by its Motion a Line ſimilar to the directive Line. But if all Lines that can be drawn from V to BC (which we may call the Sides of the propoſed Superficies) be proportionably cut, (which will be done by a Plane drawn thro' this Superficies parallel to the Plane in which BC is ſituate) the Points of Diviſion will conſtitute a Line; or at leaſt fall in a Line ſimilar to BC. For drawing any Sides VB,VD,VE,VC; and drawing the Plane GKLH parallel to the Plane BDEC, let the common Sections of the Plane VBD and Planes BC, GH, be the right Lines BD,GC; theſe ſhall be parallel. Alſo the common Sections DE, KL of the Plane VDE and the ſaid Planes will be parallel. Therefore the Angles BDE, GKL are equal. Alſo the right Lines BD,GK will be to one another, as DE and KL, becauſe each of theſe Ratio's is equal to the Ratio of VD to VK (the Triangles VDB,VKG; as alſo VDE,VKL being

16, 11. Euclid.

10, 11. Euclid.

being ſimilar) and alternatively BD : DE :: GK : KL. Therefore all the Subtenſes in GH are proportional to all the Subtenſes in BC, taking them orderly one after another, as they lie in each Line; and any two correſpondent adjacent ones in each Line have the ſame Inclination. Therefore from what has been above hinted it is manifeſt the Lines BC, GH are ſimilar. Hence alſo we may ſee that ſimilar Curves BC,GH have the ſame Proportion to one another, as the Sides VB,VG of the Superficies, lying in any one right Line. For ſince each of the Ratio's of the Subtenſes including the ſame Angles (as BD,GK, or DE, KL) are equal to the Ratio of the Sides VB,VG; all the Antecedents join'd together (that is, the whole BC) will be 12, 5. Euclid. to all the Conſequents taken together (that is, the whole GH) as VB to VG. Hence alſo ariſes this Property of Superficies generated by ſuch a Motion, *viz.* that the Parts of them, taken from the Vertex, lying between Planes parallel to BC and the ſame Sides wherever taken are ſimilar to one another; as ſuppoſe the Superficies BVC,GVH; and BVD,GVK. (Which will more evidently appear from the general Doctrine of Similitude hereafter to be explained; in the mean while this will evidently enough be manifeſt from the Similitude of Curve Lines, and their

Analogy to the Sides of the Superficies; and entirely by a like Generation of the Superficies; at leaſt it appears ſufficiently from the Similitude of the Triangles VBD, VGK; and VDE, VKL, and of all ſuch; ſince each Superficies may be looked upon as conſiſting of an Infinity of theſe Triangles.) Whence the Properties of ſimilar Superficies will agree with them. But with regard to Superficies intercepted by different ſides, in comparing theſe it is to be obſerved, they are not always proportionable to their Baſes or the reſpective Parts of the directive Line; this happening only when all the Sides of the ſaid Superficies are equal to one another, and ſo the directive Line is the Periphery of a Circle; in which Caſe the Superficies generated is a conical Superficies ſtrictly ſpeaking, being indeed that of a right Cone. But if the directive Line BC be ſuppoſed, for Example, to be circular, and the Sides unequal to one another, if BC be divided into equal Parts, and the ſides VD, VE be joyned, the Superficies BVD, DVE, EVC will not be equal to one another, but unequal moſt commonly after an inſcrutable Manner, according to the Diverſity of the included Angles and Inequality of the Sides; which is what has hitherto vex'd and hamper'd *thoſe who have ſet about meaſuring the Superficies of an oblique Cone.*

From hence it follows too that a Circumlation of this kind may be ſuppoſed to be effected in ſome ſort by ſuch a Motion of the generative Line, that every Point thereof carried parallel-wiſe, can deſcribe Lines ſimilar to the directive Line, by ſuppoſing the generative Line to be proportionally every where contracted or dilated in all the Parts thereof. As if the right Line VB be conceived to be gradually drawn after ſuch a Manner that the Point B ſhall trace out or move along the whole Line BC; the Point G carried on with a Motion parallel to BC will deſcribe the Line GH ſimilar to BC. Moreover if in like manner the Curve BC be conceived to be moved with a direct Motion along the right Line BV keeping a Situation always parallel to it ſelf; in ſuch a Manner that every Point of it does deſcribe right Lines, all meeting one another in the Point V; that is, if it be proportionably contracted in all its Parts during its whole Motion, quite to the Vertex V; by theſe Motions will the very ſame conical Superficies be produced as thoſe but now mention'd. But theſe kind of Motions are imaginary and ſuch as Nature does not admit of. Yet notwithſtanding they may ſerve to explain the Nature of ſuch ſort of Superficies, and at leaſt can be conceived as poſſible to be effected *by a divine Power.*

Again, if the directive Line in the Motion aforeſaid, be ſuppoſed every way incloſed, ſo as to comprehend ſome Figure, the Curve Superficies ariſing with this Figure for a Baſe, will include a *pyramidal* Body, or *conical* one (ſtrictly or laxly taken according to the Nature of the ſaid Figure) whoſe general Properties do ſufficiently appear from what has been ſaid, *viz.* that Planes parallel to the Baſe of it do cut off ſimilar Superficies next the Vertex, ſimilar Baſes, and ſimilar Solids too. In a Word, I ſay, whatever *Euclid*, *Apollonius*, and others, have generally given us about Cones, theſe by obſerving a due Analogy, will agree to Cones produced after this manner, and will be much after the ſame manner demonſtrated ſo to do.

But the moſt uſual manner among *Mathematicians* of producing Bodies, is that which is peculiarly call'd by the Name of *Rotation*, which is performed by ſuppoſing any Line or plane Superficies to revolve about an immoveable Line, as an Axis. As a *ſpherical Superficies* is produced from the Motion of half the *Circumference* of a Circle about the Diameter, a *Sphere* by the Rotation of the Semicircle itſelf; a *cylindrical Superficies* by the Motion of a right Line about a Line parallel to itſelf; a *right Cylinder*, by the Motion of a right angled Parallelogram about one ſide; a *conical Superficies*, by the Motion of one ſide of a right lin'd Angle about the other; a Cone it ſelf by the Rotation of a right angled Triangle about one of its Legs, and after this manner may be generated *innumerable ſolid Magnitudes with their Curve Superficies, either the Wholes, or their Parts, Fruſtums, Tubes, Rings.* The moſt chief Property of which Motion is this, that every particular Point of the Magnitude moved about does deſcribe circular Peripheries (compleatly effected, when the Revolution is quite performed, or the moveable Magnitude returned to its firſt Situation, all deſcribed at the ſame Time being ſimilar) whoſe Centers are all in the ſaid Axis, and *Radii* right Lines perpendicular to it. Or, that all right Lines ſituate in the moveable Magnitude being perpendicular to the Axis do deſcribe Circles (if the Revolution be ſuppoſed compleatly finiſhed) or ſimilar circular Sectors, I mean ſuch as are deſcribed at the ſame Time. As if, for Example, any right Line revolves about the Axis VK, by this Motion will be generated ſome Curve Superficies, made up as it were of circular Peripheries, (for I ſpeak in the Phraſe of the *Atomiſts* for the ſake of Eaſe Brevity and Perſpicuity; and I make Fig. 9.

make no Scruple of uſing their Method, becauſe of its Truth.) I ſay, the circular Peripheries AY,BY,CY,DY, paſſing thro' the Points A, B, C, D, and all the other Points which lie round about in VD, the *Radii* of which are the right Lines AZ, BZ,CZ,DZ, perpendicular to the Axis and Centers Z in the Axis. But if the Revolution be continued only ſo far, untill the Situation of VAD be in V *a d*, it will appear the Superficies generated by the Arches A*a*, B*b*, C*c*, D*d* are ſimilar to one another. In like manner, if the Plane VDZ revolves about the Axis VK, ſuppoſing an entire Revolution to be made, a Solid will be produced, conſiſting as it were of innumerable parallel Circles AY, BY,CY,DY, whoſe *Radii* (as at firſt) are AZ,BZ,CZ,DZ, and Centers Z: and ſuppoſing that the circulation leaves off in the Situation DV*d*, a Solid will be formed of the Sectors AZ*a*,BZ*b*,CZ*c*, &c. ſimilar to one another. Here we muſt not paſs by a very uſeful and neceſſary Obſervation about the manner of inveſtigating the Dimenſions of Superficies and Solids thus generated by the Method of indiviſibles, being the moſt expeditious of any, and no leſs certain and infallible when rightly applied. The very learned *Andrew Taquet* more than once objects to this Method, in his little Tract concerning *Cylindrical*

lindrical and annular Solids, and thinks to overthrow it for this Reaſon, that the Superficies of *Cones* and *Spheres* (I conceive the Quantities of them) found out by the ſame do not correſpond with the Dimenſions diſcovered and ſet down by *Archimedes*. For Example, let DVY be a right Cone, and VK the Axis, thro' all the Points of which let the right Lines ZA,ZB,ZC,ZD, &c. be conceived to paſs perpendicular to the Axis, of which to wit the right angled Triangle VKD, according to the *Atomical method*, conſiſts; and the Cone itſelf of the Circles deſcribed from theſe as *Radii*. Therefore he argues that a *conical* Superficies is made up of the Peripheries of theſe Circles, which notwithſtanding by Experience is found to be contrary to Truth, and accordingly the Method fallacious. But I reckon this to be a wrong Way of entrance into the Calculation; for when we are computing the Peripherie of which Superficies conſiſt we muſt not proceed after the ſame manner as we do with the Lines of which plain Superficies are made, or the Planes of which Bodies are formed. That is, the Peripheries conſtituting the Curve Superficies generated by the Motion of the Line VD, muſt be looked upon as equal in Multitude to the Number of Points in the ſaid generative Line: Becauſe there Fig. 10, 11, 12.

Fig. 10, 11, 12.

8

cannot be more of theſe Peripheries, than what can paſs thro' every Point of it; howſoever remote or near the Axis falls, for it is this alone according to its various Remoteneſs or Nearneſs and Poſition, that determines the Magnitude of the ſaid Peripheries. But the Multitude of Lines of which the Plane DVK is ſuppoſed to conſiſt, and of Planes of which the Solid DVY is made up, is to be rated by the Number of Points in the Axis VK; for there cannot more parallel right Lines or Planes perpendicular to VK, be contained within the Limits VK than what are equal in multitude to thoſe Points. In obſerving which *difference* (carefully to be minded) we ſhall avoid all Error, and *in my Opinion, find out the Superficies generated by the Rotation of ſuch like Curves by a Way the moſt eaſy that the Nature of theſe things admits.* For Example, if it be required to meaſure the Superficies generated by the Revolution of the Line VD about the Axis VK; we muſt conceive VD to be ſtretch'd out into (or become) a ſtrait Line, ſo as to be equal to itſelf, and let right Lines be conceived to be apply'd to all Points thereof perpendicular to VD; each in order equal to the circular Peripheries of which the Curve Superficies conſiſts, ſuppoſe AX to AY, and CX to CY, each to each, and ſo on continually. The Plane VDX made up of theſe Parallels will be equal to the ſaid *Curve Superficies*; and the Parts of this reſpectively to the Parts of that. But if inſtead of *Peripheries* be apply'd their reſpective *Radii* AZ,BZ,CZ, &c. the Space made up of theſe Lines (which doubtleſs go on with the others in a proportionable Series,) will be to *the Curve Superficies, as the Radius of any Circle to its Circumference.* Therefore if by any Way you can find the Sum of all the *Radii* paſſing thro' all Points of the generative Line (that is, if we happen to have the *Meaſure* of the *Space* VDZ) that of the Curve Superficies will be immediately obtained by it. As, ſuppoſe the *conical Superficies* DVY be propoſed, generated by the Rotation of the right Line VD about the Axis VK. To the right Line VD apply the right Lines AZ, BZ, CZ, DZ perpendicular to VD, and each of them equal to every of the *Radii* of the Circles in the Cone expreſſed by the ſame Letters. In this Caſe there will be formed a triangular *Space* VDZ, becauſe the right Lines AZ, BZ, CZ being at equal Diſtances from one another do equally encreaſe; which is altogether proper to the Ordinates of a Triangle. But the Meaſure of this

Triangle is immediately had from the given Altitude VD and Baſe DZ. And then if it be made, *as the Radius of a Circle to its Circumference, ſo is the Triangle* VDZ *to a fourth Proportional:* This ſhall be equal to the *propoſed conical Superficies.* And juſt after the ſame manner you may eaſily inveſtigate the *Superficies* of a *Sphere*, and the Portions thereof. Or ſuppoſing thoſe required, given or pre-known, any others whatever generated in this manner may be found out. But my Deſign is to inſiſt only upon generals.

Laſtly, this Geneſis of Magnitudes is rivaled by that other, ſomewhat a-kin to it, where innumerable ſimilar Lines or Figures are ſo conſtituted Parallel-wiſe about a right Line, (or indeed about any other,) that each of them have their Center in the ſaid Line, which accordingly is in the Nature of an Axis, and is ſo called. As in *oblique Cylinders* and *ſcalene Cones* the Circles ſtand about a certain Line; which accordingly is called the *Axis* of them, becauſe the Centers of the parallel Circles do all lie in it. But ſince Nature does not allow of Motions thus diſtorted (at leaſt according to the ſimple Manner of Operation we now ſuppoſe) and becauſe Magnitudes of this ſort may be other ways conceived eaſily generated, we ſhall not meddle with them. Nor indeed ſay any more of the Generation of Magnitudes by ſimple Motions.

LECTURE III.

I HAVE hitherto endeavoured to ſhew how Magnitudes may be conceived to be generated by Motions either *progreſſive* or *circumrotary*, and enumerated ſome of the chief Properties ariſing therefrom; we ſhall now do ſomething to the ſame Purpoſe, by means of compound and conſpiring Motions, in deſcribing the Effects whereof, the Velocities wherewith the ſingle Motions are performed, muſt for the moſt Part be conſider'd; tho' in *Generations* by ſimple Motions they are not in the leaſt regarded. For the ſame Magnitude may be produced with the ſame ſimple Motion, whether ſwifter or ſlower it matters not, altho' not in the ſame Time: As, for Example, the ſame *Circle* by the Rotation of a right

Line about a Point fixed in it; the *ſame Sphere* by the Motion of a Semicircle about a *Diameter*; which will ſo much the ſooner be done, as the ſaid generating Motion is ſwifter. But in a generation by compound Motion, the ſame Ways of *Lation* remaining, as the Velocity of one or ſeveral varies, there ariſes Magnitudes not only altering in Species, but Quantity too, or at leaſt perpetually differing in Poſition. As if a right Line AB be carried along the right Line AC, with a parallel equable Motion, and at the ſame Time a Point M uniformly deſcends in AB; or if at the ſame Time, the right Line AC likewiſe deſcends with a uniform parallel Motion interſecting AB as it moves in the Point M; a right Line AM will be produced by the Compoſition or Concurrence of theſe Motions. Alſo if the Motion of the right Line AB remains the ſame, as to Velocity. But if the Velocity of the uniform Motion of the Point M, or right Line AC be changed; thus indeed will the Point M arrive at *m*, or AC Interſect AB in *m*, in the ſame time; but another right Line A*m*, differing from the former in Poſition, will be deſcribed by this Motion. And if again, the Motion of the right Line AB remaining the ſame, the Motion of the Point M or right Line AC, inſtead of being uniform, be now equably Accelerate, Fig. 13.

as it is called; by ſuch a Compoſition or Concurrence as this will a *parabolick Line* AMX be formed. Or alſo another A*m*Y, (according as the Degree of this Accelerate Motion may be ſuppoſed to vary.) But if the Velocity of the ſaid Point or Line be conceiv'd to increaſe or decreaſe according to other Laws, there will thence ariſe different Species of Magnitudes. In theſe Examples, it appears that *Compoſition* and *Concurrence* of Motion do now and then amount to the ſame thing, for this Reaſon, that becauſe each of the Points of any right Line moved along with a parallel Motion, do deſcribe right Lines parallel to one another, it is the ſame thing whether any fix'd Point thereof be carried along with it, or moves ſeparately through a Line parallel to its Direction; as whether the Point M fixed in AC be carried along with it, or freely moves with the ſame Velocity along the right Line AB. But it oftentimes happens that the *Generation* of *Magnitudes* is not ſo eaſily ſhewn by either of theſe Ways. For let the right Line AB revolve equably, (that is, ſo as to make equal Angles in equal Times) and at the ſame Time let the Point M move from A in the right Line AB with an equable Motion too; by this Compoſition of *Motions*, will be produced a certain Line, *viz.* the *Spiral* of *Archimedes* Fig. 14.

(for we have deſignedly made Choice of *ſuch Examples*, as may gradually render the Nature of the moſt eſteem'd *Magnitudes* familiar to thoſe who are not much verſed in theſe things) the Generation of which in my Opinion, cannot be evidently and conveniently enough explained by any Concurrence of moveable Lines, *&c.* *viz.* after ſuch ſort that the *Ratio* or Quantity of thoſe Motions, or of the Lines or Angles determining the Quantity of them can be known. Indeed it might be generated by the Concurrence of the parallel Motion of the right Line AC; or by the circular Motion of a right Line BA about any Center B, concurring with the aforeſaid regular Motion about the Center A. But it would have been difficult then to have found out the Quantity of the right Lines AM, A*m*, or Angles ABM, AB*m*. On the contrary, if the right Line BA revoles uniformly about the Centre B, and at the ſame time a right Line AC be moved along AB parallel-wiſe with a uniform Motion, the continual Interſection of the right Lines BA, AC thus moving, will beget a certain Line, called a *Quadratrix*, whoſe Generation cannot be ſo evidently expediated or explained by the ſaid Compoſition of Motions ſtrictly ſpeaking. Indeed it may be generated by the ſtrait Motion of any Fig. 15.

Point M in AB carried along parallel-wiſe to the Line AB firſt propoſed; or alſo by ſuch a Point in AC moved parallel-wiſe; or by the Motion of a Point in AB, (moving about B or A) going directly on from A towards B, or from B towards A. But whatever Compoſitions of Motions of this ſort be ſuppoſed, it is very difficult to get the Quantity of the right Lines AM or BM, or Angles BAM, or ABM, or AMB, or of any Magnitudes determining theſe Motions, or their *Ratio* to one another. Which is the chief Reaſon why I make a Difference between *Compoſition* and *Concurrence* of Motions: Becauſe the Generation of Magnitudes is ſometimes more eaſy to be explained one Way, ſometimes another. But I now proceed to expound them more diſtinctly. And firſt of Compoſition. And ſince a Motion may be conceived two ways to be compounded; either as the aggregate of ſeveral Motions, or as partaking of ſeveral; we ſhall handle the latter, which perhaps cannot be ſet in a clearer Light than in the ſecond Book of the *Principia* of the excellent Philoſopher *Deſcartes*. His Words are, "Altho' each Body of any Kind ſoever, "has but one Motion peculiar to itſelf, "becauſe it is conceived to proceed only "from the Bodies to which it is join'd "or contiguous, which are at reſt; yet

" the ſame may partake of innumerable
" other Motions: If it be a Part of other
" Bodies having other Motions. As, if a
" Perſon walking in a Ship, carries a
" *Watch* in his Fob; the Wheels of this
" Watch will move with one Motion on-
" ly peculiar to themſelves: But they will
" partake of another Motion alſo, in as
" much, as being contigious or join'd to the
" Perſon who is walking, they both to-
" gether make up one Part of Matter;
" and of another, in as much as being join'd
" to the Ship fluctuating in the Sea; and
" of another ſtill, ſo far as being join'd to
" the Earth it ſelf, provided the whole
" Earth moves. And the Wheels of the
" Watch have all theſe Motions in reality.
" But becauſe ſuch a Number of Mo-
" tions cannot eaſily be conceived, or in-
" deed all of them in it known, it is
" ſufficient to take Notice of that one
" only, which is peculiar to every Bo-
" dy whatſoever. Which peculiar Mo-
" tion of every Body may be conſidered
" as ſeveral, as when we diſtinguiſh
" two different ones in the Wheels of
" Coaches; one about their Axle-tree,
" and the other ſtrait according to the
" length of the Way along which they
" move. But that therefore ſuch Mo-
" tions are not really diſtinct, appears by
" this, that every Point of the Body which
moves

“ moves deſcribes but one Line. Nor is
“ it of any Moment, becauſe this Line may
“ be often very crooked, and ſo ſeems to
“ be generated by ſeveral differentMotions;
“ becauſe we may imagine in the ſame
“ manner any Line, even a right one,
“ which is the moſt ſimple of all, to be
“ generated by many different Motions, as
“ if the right Line AB be moved towards
Fig. 16. “ CD, and at the ſame time the Point A
“ be moved towards B, the right Line
“ AD, that this Point will deſcribe, does
“ no leſs depend upon two ſtrait Motions,
“ from A to B, and from A B to C D,
“ than a Curve Line, which is deſcribed
“ by any Point of the Wheel, depends
“ upon a right and circular Motion. And
“ accordingly, altho' it is very neceſſary
“ to diſtinguiſh one Motion in this man-
“ ner into ſeveral Parts for a more eaſy
“ Perception thereof; yet, ſtrictly ſpeaking,
“ we are to allow but one Motion to any
“ Body.” Thus far *Cartes*. Whoſe meaning in ſhort is this; ſince whatever Magnitude, which is any how join'd to another, does ſo partake of its Motion, as to be in ſome meaſure determined by it, with regard to its Situation; that becomes, as it were a part of the Compoſition of this Motion, as will evidently appear from Examples hereafter laid down. But Motions may be compounded after this manner;

progreſſive with *progreſſive*; *progreſſive* with *circumrotary*; and *circumrotary* with *circumrotary* ones: I ſay, they may be *compounded* and *uncompounded* innumerable Ways; all of which ſince it is impoſſible to reckon up, and difficult to underſtand, and more ſo, to explain ſuch as are not regular, we ſhall only regard ſome of the chief, that are the moſt Uſeful and Eaſy to explain. Among which are principally ſuch as are compounded of *direct* and *parallel*, of *direct* and *circumrotary*, and of ſeveral *circumrotary* Motions; eſpecially thoſe that the ſimple Motions of which they conſiſt, are all or ſome of them equable. *For Uniformity is not only neceſſary in the State and Church; but exceedingly requiſite in Arts and Sciences. Strait Motions*, among which I reckon parallel Motions directed by a ſtrait Line, do deſervedly take place of all others, as exceeding in ſimplicity, acceptable and agreeable to Nature, and more uſeful than any others whatſoever. Neither indeed is there any kind of Magnitude, (no Line, no Superficies, no Solid,) but what may not be conceived to be generated by ſtrait Motions. I ſay, every Line lying in one Plane may be produced from the parallel Motion of a right Line and Point in it; every Superficies from the parallel Motion of a Plane and a Line in it; (*viz.* of ſome Line generated in the manner but now mention'd by ſtrait Motions:) And conſequently any Line too may be produced by direct Motions in a Curve Superficies. Alſo a ſolid Body can be ſuppoſed to be figur'd after the ſame ſort, in as much as it ariſes from Superficies, and is terminated and circumſcribed by Superficies thus generated. But becauſe the *Generation* of the moſt Part of *Curve* Superficies, hitherto received in *Mathematicks*, and that of Lines in them not lying in the ſame Plane, may be explained more conveniently by other Methods, nor can I ſay any thing offers to me relating to them worthy of Notice, we ſhall only conſider the Generation of Lines all in the ſame Plane, produced by direct or parallel Motions. And indeed there is no one of theſe but may be generated by the parallel Motion of a right Line, and a Point at the ſame time moving along the ſame: But theſe Motions ought to be tempered in ſuch a manner as the particular Nature of the Curve requires; nor may any regard be had how variable a Velocity you attribute to one of the Motions, provided that of the other be duly accommodated to it. As for Example, Fig. 17. if the right Line ZA be always carried along the right Line A Y parallel to itſelf with any equable or unequable Motion (increaſing, decreaſing, or varying in Velocity according to any manner imaginable)

ble) and at the ſame time any Point M be moved along it, ſo that the Motions of the Point in every part of Time, be proportionable to the Motions of the right Line; there will be a *right Line* produced. That is, if it be always AB : AC : : BM : C*m*. or AB : MX : : AM : X*m*. (ſuppoſing MX parallel to AC) It is manifeſt all the three Points A, M, *m* do lie in one right Line. For a Property of a right Line demonſtrated in the 6th Book of the *Elements* of *Euclid*, is, that, right Lines apply'd parallel to it, proportional to their Diſtances from an aſſigned Point in it do terminate in a *right Line.* Alſo if theſe Motions be ſo tempered, that taking any Line D, the Rectangle under the Difference between the Lines D, BM (moved through by the moveable Point in the right Line AZ) and BM, has always a given *Ratio* to the Square of AB (moved through by the Line AZ in the ſame Time) an *Ellipſis* or *Circle* will be deſcribed: The latter when the propoſed *Ratio* is a *Ratio* of Equality, and the Angle ZAY a right Angle; and the former, when it is otherwiſe; the Line D being one of the *Diameters*, ſituate in the Line AZ firſt propoſed, extended from the Vertex A towards Z. But if they be ſuch, that the Rectangle under the Sum of the Lines D, BM, and BM always is in the ſame Proportion to the Square of AB: an *Hyperbola* will be produced by this compound Motion; being *equilateral*, when the ſaid *Ratio* is a *Ratio* of equality, and the Angle ZAY right: But if not, it will be of another Kind, according to the Quantity of the aſſign'd *Ratio*, whoſe *tranſverſe Diameter* will be equal to D, ſituate in ZA firſt laid down, and extended from the Vertex A the contrary Way to Z, the *Parameter* being determinable from the given *Ratio*. Alſo if the Rectangle under D and BM which is moved over has perpetually the ſame *Ratio* to the Square of AB, there will be a *parabolick Line* deſcribed, whoſe *Parameter* may be defined eaſily from the Quantity of the right Line D, and that of the given *Ratio*. And in the firſt of theſe Caſes, if the tranſverſe Motion along AY be ſuppoſed uniform; the deſcending Motion through AZ ſhall be uniform too: In the ſecond and third, if the Motion along AY be uniform, the Motion deſcending will perpetually encreaſe; and ſuppoſing the ſame thing with regard to the laſt Caſe, wherein the *Parabola* is deſcribed; the Point M will perpetually encreaſe equably in Velocity. Much after the ſame manner may any other Line be conceived to be generated by ſome ſuch like Compoſition of Motion. But that we may arrive at length to the intended purpoſe;

poſe; come on, and let us ſee what Advantage will accrue to *Mathematical Matters*, by ſuch a ſuppoſed Generation of Lines. In order to which, that our Buſineſs may be as ſimple and clear as poſſible, we ſuppoſe one of theſe Motions, *viz.* of the right Line keeping its paralleliſm, to be always uniform, and then lay hold upon the general Affections of the Lines produced flowing from the general Differences of the other Motion, with regard to Velocity. But this ſhall be the Work of the next Lecture.

LECTURE IV.

Fig. 18.

WE propoſe to find out and explain the Affections of Lines ariſing from ſuch a Compoſition of Motions, as has been already deſcribed. In order to which, for Method's ſake, I again ſay, if a right Line AZ moves perpetually Parallel to itſelf, with a uniform Motion along the right Line AY, and at the ſame time a Point M be carried in it uniformly with any Velocity whatſoever; the Point M will deſcribe a right Line; for take any two Places AZ, of the moveable Line, *viz.* at B and C: Then becauſe the Motion through AY is ſuppoſed uniform, the Spaces AB, AC moved through will be to one another as the *Times*: But becauſe of the uniform Motion of the Point M, right

Lines BM,CM will be alſo as the ſame Times. Therefore, it is AB : AC : : BM : CM. Wherefore it is manifeſt the Points A,M,M do lie in one ſtrait Line. And by a like Demonſtration, the ſame Thing is evident of any Points whatſoever, whereat the Point M during its whole Courſe ſtands. Secondly, if the Point M be ſuppoſed to move with a continual increaſing Motion, according to any manner of Velocity, whether regular or irregular, I affirm *the general Properties which we ſhall ſubjoin of the Lines generated do follow from this Suppoſition.* (It being doubtleſs the better Way to gain all the Properties together, from one common Generation, which agree to any one Kind of Lines, than to demonſtrate ſingly one by the other, as is moſt commonly done.) But in the mean while obſerve, for the ſake of Brevity, that I ſhall call hereafter the uniform Motion of the right Line AZ along AY, the *tranſverſe Motion*, and the Motion of the Point moving from A in the Line AZ, a *deſcenſive* or *deſcending Motion*, regard being had to the Figure exhibited. Alſo becauſe of the Uniformity of the Motion along AY and the Parallels to the ſame, it may repreſent the Time of the Motion, and the Parts of the one, the Parts of the other. Fig. 19.

We now proceed to the Explanation of the ſaid Properties.

Fig. 19. 1. The Line generated after this way, *viz.* by a uniform tranſverſe Motion, and a deſcenſive Motion continually increaſing, will become curved in all its Parts. For take any three Points M,N,O in it, thro' which draw BZ, CZ, DZ, parallel to AZ, and thro' the Points M,N, draw the right Line MNK. Now becauſe the right Line MN is generated by the compound Motion of the *tranſverſe Motion* along BC, or MG parallel to it, and the *deſcending Motion* along AZ, being both equable or uniform, and the tranſverſe Motion through MG is entirely the ſame with the tranſverſe Motion, whereby the propoſed Line MNO is deſcribed; it is evident the Velocity of the uniform deſcenſive Motion, generating the right Line MN is leſs than the Velocity, which the deſcending Motion in like manner deſcribing the Line MNO, has in the Point N (for unleſs this Motion be not ſwifter than that, ſince it is ſuppoſed to increaſe continually, it would be ſlower than that in the whole Time of Deſcent thro' GN, and ſo would not paſs over an equal Space in the ſame Time, and both together arrive at the Point N) Therefore this unequal accelerate Motion continu'd throughout the Time of the uniform Motion CD (whereby the Line NO is generated) meaſures out a greater Space than the deſcending uniform Motion, whereby MN, continued to K, is deſcribed in the ſame Time CD: (for it is plain a greater Space will be moved thro' in the ſame Time by a greater Force encreaſing, than by a leſs that increaſes not at all.) Wherefore the Line HO is greater than HK; and ſo the three Points M, N, O do not lie in the ſame right Line; and ſince this is true of any other three Points of the Line MNO, it is moſt evident the ſame can be no where ſtrait, but bent or curved in every part thereof.

2. Hence we have got the following *Corollary.* The Velocity of the deſcenſive uniform Motion, whereby is deſcribed any Subtenſe (as MN) of the Curve MNO (*viz.* the common tranſverſe Motion whereby it and its Arch are deſcribed, being uniform) is leſs than the Velocity, which the deſcenſive increaſing Motion has at the common Term N of both.

3. Any *Subtenſe* (as MO) of this Curve, falls entirely within its Arch, towards AZ, and continu'd falls beyond the Line MNO. For if any Point N be taken in the Arch MO, and the right Lines MN, NO be joyn'd, it follows the whole MO lies within the right Lines MN, NO, and ſo within the Curve MNO. But the

whole, if continued out, falls without the Line MNO, becauſe it meets the ſame no where elſe, as has been juſt now demonſtrated.

Elem. 3. 2. *Apoll.* 1. *Seren.* 1. 8.

Euclid has particularly demonſtrated this of the Circle, *Apollonius* of Conick Sections, and *Serenus* of Cylinders.

4. It is manifeſt the propoſed Curve is Convex or Concave the ſame Way (Convex towards the upper or exterior Parts AY, and Concave inwards or downwards towards AZZ.) For no more is underſtood by a Curve's being Convex or Concave the ſame way, than that a ſtrait Line can cut it but in two Points; nor does *Archimedes*'s Definition of a Line Concave the ſame Way, which we find in the Beginning of his firſt Book of the *Sphere* and *Cylinder* imply any thing elſe. It is evident, for Example, that a right Line MN cutting the Curve MNO in two Points M,N, if it meets the ſame again, as ſuppoſe in K, the Curve MNO muſt be reflected, and bent up towards AY, which is what we have juſt now ſhewn cannot happen. Therefore the ſaid Line is Convex, or elſe Concave the ſame Way.

5. It is moſt Evident, that any right Lines (as BZ, CZ) parallel to the *generative* Line AZ, do cut the propoſed Curve. (Suppoſing them contained within the Li-

mits of the Motion through AY; becauſe the Curve is conceived to be deſcribed by any one of theſe indefinitely extended.) I add that any one of theſe Parallels cuts the Curve in one Point only. This appears, becauſe the *generative right* Line AZ continues but a Moment of Time in any one Situation, as BZ; inſtantly arriving at, and leaving BZ; and beſides, there is but one Point M of the Curve in BMZ, all the reſt being in the Parallels to BZ. Therefore it is evident, that BZ cuts the Curve only in one Point. *Apollonius* has given us ſeparate Demonſtrations of this, in the *Parabola* and *Hyperbola*; and *Archimedes* in the Sections of *Conoides*. *Apollon.* 26. 1. *Archim.* 16, 9. *Conoid.* and *Spher.*

6. In like manner it is evident that any Parallel (as PG) to AY touches the Curve in one Point only. That it ſhall once meet the ſame (ſuppoſing it contained between the Limits of the Deſcent through AZ) is evident, becauſe the moveable Point conſtantly deſcending, by indefinite Advances, will one time or other paſs over it when indefinitely continued; and yet nevertheleſs can continue in it but one Moment of Time. See *Apollonius*'s Demonſtration of this. *Prop.* 19. *Book* 1.

7. All right Lines ſubtending the Curve, if continu'd out, will meet AZ and the Parallels to it.

For that any Subtenſe, as MN, will meet ſome one of the Parallels, as BR, *viz.* there where it cuts the Curve, is very plain; becauſe the whole Curve is deſcribed by the parallel Motion of the ſaid right Line. And ſo ſince it meets one, it will meet all; for that right Line parallel to any one of ſeveral Parallels, is parallel to them all, as is demonſtrated in the firſt Element.

Apollonius has thought fit to demonſtrate this ſeparately of the *Parabola* and *Hyperbola*. *Prop. 22. Book 1.*

8. It is evident in like manner that all right Lines touching the Curves, excepting one only at the End of the recurring Line, (Vide 18 of this) will meet the ſaid Parallels. Which *Apollonius* too has demonſtrated of *Conick* Sections in the the 24*th* and 25*th* Propoſitions of his firſt Book.

9. Moreover any right Lines interſecting A Z (below the Point A, or above the Limit, if there be any of the deſcenſive Motion) will cut the Curve.

Becauſe all right Lines that do cut the Parallels to AZ, infinitely produced muſt needs cut the Curve.

Apollonius has given very laborious Demonſtrations of ſuch like Properties of the *Conick Sections.*

Apoll. 27,28. 1.

10. Alſo it is manifeſt the Ordinates to the right Line AY, parallel to AZ (which may not improperly be call'd the verſed Sines of the propoſed Curve) have a leſs Ratio to one another (*viz.* by comparing the leſſer with the greater, or putting the leſſer in place of the Antecedents,) than the reſpective Parts of AY, moved over in the ſame Times, (which I make no ſcruple of calling right Sines of the propoſed Curve. That is, BM has a leſs Ratio to CN, than AB to AC, or BM to CF. Becauſe CN $\sqsubset$ CF; you will find this (moſt commonly in Authors) demonſtrated ſeparately of Circles and other Curves.

In order to what follows, we muſt obſerve, if the tranſverſe Motion of a right Line be conceived to be backwards, *viz.* from D towards A through DA, beginning at any Point, as O, of the Curve, and the Velocity of the aſcending Motion of the generative Point decreaſes continually from O after the ſame Manner, as it increaſed during its Deſcent to O; the ſame Curve will be deſcribed, provided the tranſverſe backward Motion be equal to the forward one. This needs no Proof, for it is but the ſame Motion conſider'd inverſly.

11. Let us ſuppoſe the right Line TMS to touch the propoſed Curve in the Point M, (I mean touch it ſo as not to cut the ſame) and let this Tangent meet the right Line AZ in T, and thro' M draw the right Line PMG parallel to AY: I ſay, the Velocity of the deſcending Point, (deſcribing the Curve by this Motion,) which it has at the Point of Contact M, is equal to the Velocity, by which the right Line TP is deſcribed with a uniform Motion in the ſame Time, whereby the right Line AZ is moved thro' AC or PM, (or which amounts to the ſame thing; I ſay, the Velocity of the deſcending Point in M, is to the Velocity with which the right Line AZ moves, as the right Line TP to PM.) For take the Point K any where in the Tangent, and thro' the ſame draw the right Line KG, meeting the Curve in O, and the Parallels AY, PG in the Points D and G. And becauſe the Tangent TM is conceived to be deſcribed by two uniform Motions; the one of the right Line TZ carried thro' the Parallels AC or PM, and the other of the deſcending Point from T through TZ; and one of theſe Motions through AC or PM is common, or the ſame with that whereby the Curve is deſcribed; Fig. 20.

when TZ is in the ſituation KG, AZ will be in the ſame. Therefore when the Point deſcending from T is in K, the Point deſcending from A will be in the Interſection (O) of the Curve and right Line KG (for it has been ſhewn already that KG cannot cut the Curve any where elſe.) But the Point O is below K, becauſe the Tangent falls quite without the Curve. Now if the Point K be ſuppoſed below the Point of Contact towards T, becauſe then OG is leſs than KG, it is evident the Velocity of the deſcending Point, whereby the Curve is deſcribed, in the Point O of the Curve, is leſs than the Velocity of the uniform deſcending Motion, wherewith the Tangent is effected; becauſe *that* always encreaſing makes a leſs Space in the ſame Time (repreſented by GM) than *this* which does not encreaſe at all; but perpetually continues the ſame; the former (to wit) paſſing over the right Line OG, and the latter, the right Line KG. But on the contrary, if the Point K falls above the Point of Contact towards S; becauſe OG is then greater than KG, it is evident the Velocity of the deſcending Point, (whereby the Curve is deſcribed,) in the Point O, is greater than the Velocity of the uniform deſcending Motion, whereby the Tangent is effected; becauſe the former Motion continually decreaſing in the ſame Time through OM, makes a greater Space OG, than the latter (not at all encreaſing, but continuing perpetually the ſame) does, *viz.* the Space KG. Therefore becauſe the Velocity of the Point deſcribing the Curve in any Point of the Curve, below the Point of Contact towards A, is leſs than the Velocity of the Motion through TP; but in any Point of the Curve above the Point of Contact, is greater than the ſame; it is manifeſt in that very Point it is equal to it. W. W. D.

12. The Converſe of this, *viz.* If the Velocity of the Point deſcending from A in any Point M of the Curve, be equal to the Velocity, whereby the Point T moving uniformly, ſhall deſcribe the right Line TP in the Time PM or AC (or if the Velocity of the deſcending Motion to M, be to the Velocity of the tranſverſe Motion, as TP to PM:) the right Line TMS will touch the Curve AMO in the Point M. May be Demonſtrated after the ſame manner, and that briefly, this,

Take any Point K, in the right Line TS, and draw KG parallel to AZ. Becauſe the Velocity of the aſcending Point deſcribing the Curve, always decreaſes towards A, from M to O, and (by ſuppoſition) the Velocity of the Point generating the right Line M T equal to it does not decreaſe, and the Time MG is common, the Space GO ſhall be leſs than the Space G K; and ſo the Point K will fall without the Curve. Alſo becauſe the Velocity of the deſcending Point, whereby the Curve is generated, continually increaſes the other Way from M towards O; but the Velocity equal to it, wherewith the right Line MS is generated, does not increaſe from M to K, and the Time MG is ſtill the ſame, it is plain the right Line GO exceeds the right Line GK; and ſo the Point K falls above the Curve. Therefore it is manifeſt all Points of the ſaid right Line do lie without the Curve, and accordingly it will touch the Curve. W. W. D.

13. Hence it immediately follows that *one right Line only does touch Curves of this ſort in one Point.*

For ſuppoſe the right Line M T to touch the Curve AMO in the Point M; and, if poſſible, let ſome other right Line MX touch it too. Then each of the right Lines XP, TM, will be deſcribed at the ſame Time, with the ſame Velocity, (*viz.* that which is equal to the Velocity of the Point deſcribing the Curve acquir'd at the Point M) And ſo the right Lines XP, TP will be equal; the whole and the Part, which is abſurd. Therefore

Euclid. 16,17. 3. *Apoll.* 32, 33, 34, 35, 36. Book 1.

fore no Line but MT is a Tangent. *Euclid* has demonſtrated this particularly of the *Circle*, *Apollonius* of the *Conick Sections*, and *others* of other Lines.

Hence we gain a conſiderable *Advantage*, all under the ſame Trouble, relating to the *Demonſtrations of inverſe Propoſitions concerning Tangents, viz.* If the Angle PMT be determined (or any other which a right Line given in Poſition makes with a Tangent at an aſſigned Point of the Curve.) Or if the Quantity of the right Line PT be given (or of any other the like Line drawn from a given Point in a right Line given in Poſition, and intercepted by the Tangent) we can by means thereof expreſs the Tangent. And contrariwiſe, if the Situation of the Tangent be given, the Quantity of thoſe Angles and Lines will be had, and ſo much Pains that many have beſtowed in demonſtrating ſuch inverſe Propoſitions may be ſpared. And this too is ſo much the more worthy of Obſervation, becauſe ſome of theſe inverſe Propoſitions are a great deal ſooner found out, and eaſier demonſtrated than others. Specimens of which I could readily give you, but that I have no Inclination to wander further.

14. From what has been ſaid it follows, that the Velocities of the deſcending Point, in any two aſſigned Points of the

Curve are to one another reciprocally in a Proportion compounded of the Ratio's of the Ordinates drawn from theſe Points to the right Line AZ, *(viz.* of the Parallels to AY) and of the right Lines intercepted by the Tangents to theſe Points and the ſaid Ordinates, or the Ratio of the Velocities is equal to the Ratio of the Ordinates taken from the Ratio of the Intercepts.

Thus if two right Lines MT, NX Fig. 21. touch the Curve in the Points M, N, meeting ZA continued out in the Points T and X; and the Ordinates NP, NQ be drawn parallel to YA, the Proportion of the Velocities at the Points M,N, is compounded of the Proportion of TP to PM, and QN to QX. For the Velocity in M to the uniform Velocity through AY is as TP to PM; and that uniform Velocity, is to the Velocity in N, as QN to QX. Therefore the Velocity in M, to the Velocity in N, is compounded of theſe two Ratio's TP to PM, and QN to QX. We may obſerve, if FE be drawn from the Point of Concurrence of the Tangents parallel to AY; it ſhall be TE : XE = TP : : PM + QN : QX.

MLP

15. We ſhall here inſert by the Way a general Solution of that *Problem*, which Fig. 22. *Galilæo* eſteemed ſo much, and imploy'd ſo much Time about, and which *Torricellius*

cellius declared he had happily and ingeniouſly diſcovered, flowing very eaſily from what has been ſaid. The Thing is thus propounded by *Torricellius*, (for I have not *Galilæo* at Hand) any *Parabola* being given, whoſe Vertex is A; *it is required to find ſome Point aloft*, *from whence if a heavy Body falls to* A; *and the Impetus here received*, *be (at that Point) turn'd into an Horizontal one*, *the propoſed* Parabola *may be deſcribed.* (Obſerve that the deſcenſive Motion deſcribing the *Parabola*, is conceived to begin from A, and not from the Point aloft.) The *Problem* is no other than this, according to *Galilæo*'s Suppoſition, to determine the particular Velocities of the uniform horizontal or tranſverſe Motion, and an equal increaſing deſcenſive one, by the Compoſition whereof the given *Parabola* is conceived to be deſcribed. But we will perform the thing generally, whatever be the Law of the encreaſing deſcenſive Motion; and ſhall ſubjoyn the particular Caſe of the *Parabola* as an Example. In the right Line AZ, which doubtleſs is a Diameter of the Curve, find ſome Point as P, from which if the Ordinate PM be drawn, and the Tangent MT, meeting the right Line AZ in T; the intercepted Line TP may be equal to PM: If then, in ZA continued out, you take the right Line A S=A P, I ſay, the thing is done.

For becauſe S A = A P, the moving Body deſcending from S to A, may be conceived to have the ſame *Impetus*, as in deſcribing the Curve from A to P, (for the Motion of the increaſing Velocity is the very ſame in both.) But that *Impetus*, is equal to the *Impetus*, whereby the moving Body deſcending from T with an equable Motion moves through the right Line TP in the ſame Time, as the right Line AZ carried along uniformly, and by that Motion conſpiring in the Deſcription of the Curve, moves thro' the right Line P M.

Therefore becauſe T P, P M are equal by Conſtruction, and ſo the Velocities of the Motions performed together, equal; the deſenſive Motion in P or M will be equal to the tranſverſe Motion deſcribing the Curve, that is, the Velocity of the Motion from S to A in the Point A is equal to the ſame. Therefore the Point S is the ſame that ought to be found, and the Thing propoſed done. For Example, Let there be a *Parabola* conceived to be generated by a uniform horizontal Motion, and an equable accelerate one. Then the Point P may eaſily be found out by *Analyſis* thus. Let the right Line R be the *La-*

tus rectum of the given *Parabola.* Then by the Nature of the *Parabola*, R × AP = PMq = TPq (from the Suppoſition of our general Method.) Alſo from a noted Property of the *Parabola* it is TPq = 4 APq. And ſo R = 4 AP; or $\frac{1}{4}$ R = AP = SA, which is what *Galilæo* has determined. But in this Caſe the Points T, S do Coincide. Again, if heavy Bodies be ſuppoſed to deſcend with an increaſing Velocity in the *triplicate Ratio of the Times*, and ſo ſuch a Motion of them compounded with a uniform tranſverſe Motion will deſcribe a *cubical Parabola*, and let R be the *Parameter* of this Curve; then will SA be = $\sqrt{\frac{Rq}{27}}$ for by the Nature of this Curve Rq × AP = PM Cub. And (by this general Rule) PM = TP, and ſo PM Cub. = TP Cub. And becauſe in theſe *Parabola*'s the Intercept of the Tangent, or the Sub-tangent is always triſected by the Vertex, (*viz.* AP = $\frac{1}{3}$ TP ſo is TP) Cub. is = 27 AP Cub. Therefore it ſhall be Rq × AP = 27 AP Cub. And ſo Rq = 27 APq, or $\frac{Rq}{27}$ = APq = ASq. we proſecute the thing propoſed in other Caſes, after the very ſame Manner. I am of Opinion, that not only ſeveral ſuch like Propoſitions of *Galilæo*, relating to this ſubject, may ſome how or other be deduc'd from hence, but made more general too, or extended, ſome to all Curves. But I ſhall forbear mentioning any thing elſe on this matter, being content with the *Specimen* before us, brought in here by the Way, and proceed on to other things depending upon thoſe aforeſaid.

16. If a *plain Superficies* be apply'd to a right Line, all and every of the Parts whereof intercepted by parallel Ordinates to that Line, be proportional to right Lines apply'd to AY ſimply divided *(viz.* parallel to AZ.) The Proportion of this Superficies to a Parallelogram of the ſame Height and Baſe, will follow the Proportion of A P, T P, intercepted by the vertical Point P, and the Tangent. As ſuppoſe to *a d* the plane Superficies *a d m* be apply'd, and A D being any how divided in the Points B,C, and in like manner the right Line *a d* in the Points *b*, *c*, it ſhall be, as BM to CM, ſo is the Superficies *b a m*, to the Superficies *c a m*, and this univerſally happens ſo in whatever Compariſons you make; *the Parallelogram a d m p being compleated, it ſhall be as the right Line* AP *to the right Line* T P, *ſo is the* Fig. 23, 24.

Superficies a d m to the Parallelogram a d m p.

For if the right Line *a d* be conceiv'd to expreſs the common Time, wherein the right Line A D is paſs'd over, by an equable Motion, and the right Line D M by a continu'd accelerate Motion; the right Line *d m* will expreſs the greateſt Velocity of this definitive Time, which the deſcending Point has in the loweſt Point M of the Curve; that is, the Velocity whereby the right Line T P is uniformly run over in the ſame Time; wherefore (by what has been already ſhewn) the Parallelogram *a d m p* will repreſent the ſpace, which is uniformly deſcribed by that ſame Velocity continuing throughout the whole Time *a d*, *viz.* that Line TP. Therefore becauſe (by the Condition of our Hypotheſis) the Figure *d a m* does repreſent the right Line DM or AP; it ſhall be, as the Figure *d a m*, is to the Parallelogram *a dmp*, ſo is AT to TP; and ſo that proportion being any how known. This muſt be ſo too, and contrariwiſe. The thing will become more evident by one or two Examples. Let the propoſed Curve be the common *Parabola*, wherein the right Lines BM, CM are to one another, as the Squares of AB, AC; that is, as the Squares of *a b*, *a c*. Therefore if the Figure *a d m* be a Triangle,

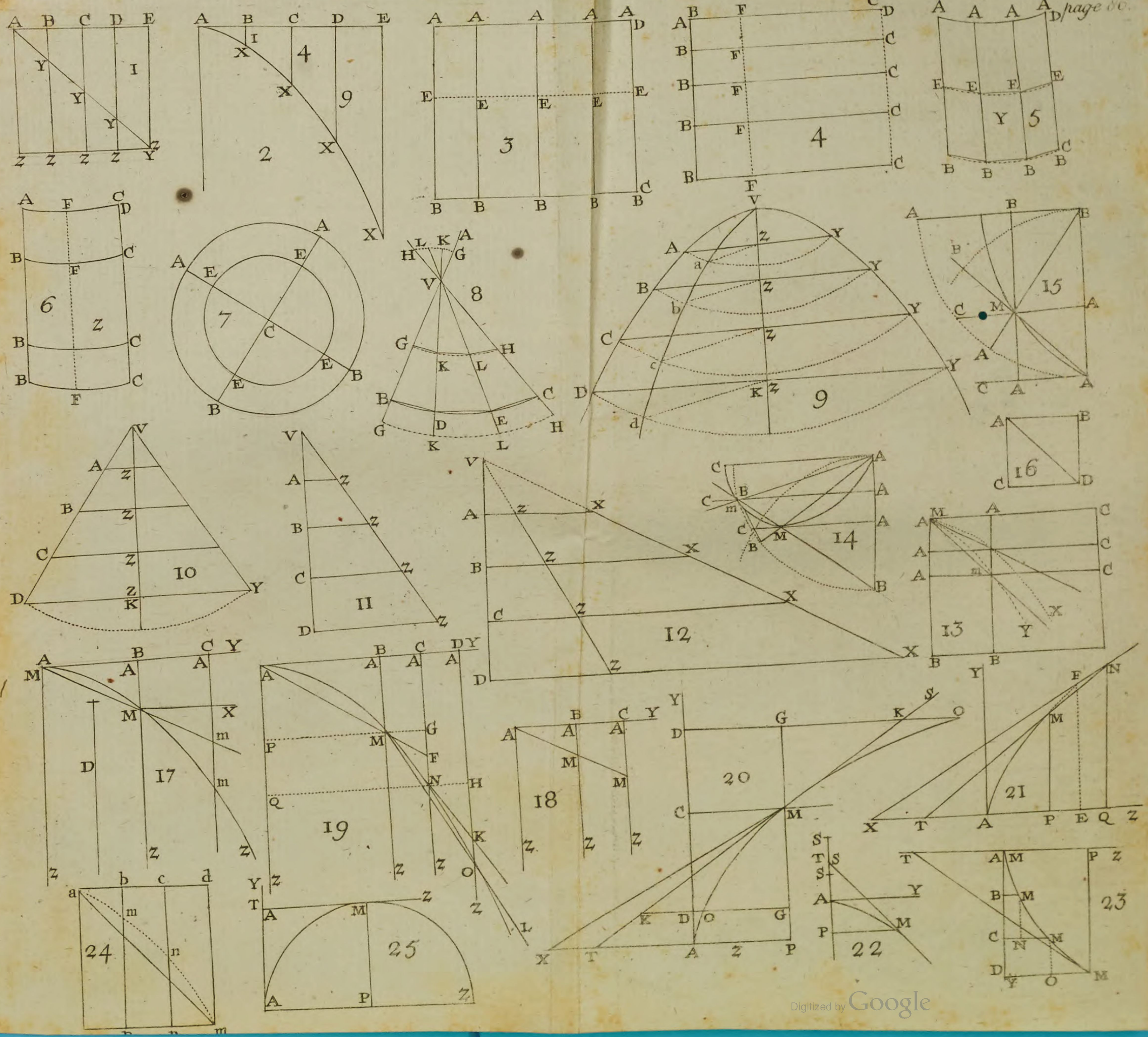
angle, it will excellently well anſwer our purpoſe. For this being ſuppoſed, the Triangles *b a m*, *c a m* will be always proportional to the Squares of *a b*, *a c*; that is, to the right Lines B M, C M. But becauſe the Triangle *d a m* is one half the Parallelogram *d a m p*; the right Line A P will be alſo one half the Line TP. Which is a well known Truth demonſtrated *in the Conick Elements.* Now let the Curve AMM be a *cubick Parabola*; and becauſe in this the right Lines BM, CM are to one another as the Cubes of the right Lines A B, A C, that is, as the Cubes of the right Lines *a b*, *a c*, and if the Superficies *a d m*, be the Complement of a Portion of one half the common *Parabola*; the trilineal Figures *a b m*, *a c m* ſhall be proportional to the Cubes of the right Lines *a b*, *a c* (as is demonſtrated by *Pappus*, and others, and may eaſily be deduced from *Archimedes*'s *Dimenſion* of the *Parabola*.) And ſo the common *Parabola* is very rightly adapted to our purpoſe; and becauſe it is elſewhere evident that the Figure *a d m* is one third Part of the Parallelogram *a d m p*; A P ſhall alſo be one third of T P, according to the directions of the Rule now aſſign'd, which Concluſion is ſufficiently agreed to by *Geometricians.*

Fig. 23,24 17. A way like to this of comparing the ſaid right Lines AP, TP is contain'd in the following *Theorem.* If to all and every of the Points of any right Line be apply'd parallel right Lines, proportional to the Differences of the Ordinates to the right Line AD divided after the ſame Manner, the Proportion of the plane Figure hence ariſing to a Parallelogram of the ſame Altitude, and put to the ſame Baſe, will give the Relation of AP, TP. As if the right Lines AD, *a d* be alike divided (*viz.* into an indefinite Number of equal Parts) and the right Lines *b m*, *c m*, *d m* be proportional to the right Lines BM, NM, OM (being the Differences of right Lines apply'd to AD beginning at the Point A.) It ſhall be as the Figure *a d m* is to the Parallelogram *a d mp*, ſo is AP to TP. For ſince any one of the Ordinates to AD, ſuppoſe, for Example, DM, is equal to all the Differences of thoſe leſs than itſelf, (*viz.* to BM, NM, OM) and the trilineal Figure *a d m* is made up of right Lines *b m*, *c m*, *d m* increaſing in the ſame Proportion; and likewiſe the right Line CM is equal to BM, NM; and the correſpondent trilineal Figure *a c m*, as it were, conſiſts of the Parallels *b m*, *c m* increaſing after the ſame Manner, and ſince this always happens to be ſo; it is entirely evident that the trilineal Figures *a d m*, *a c m*, *a b m*, are proportional to the right Lines D M, C M, B M; and accordingly this Method falls again into the former; they being really not different. But here obſerve the right Lines *b m*, *c m*, *d m* do repreſent the Velocities which, the moveable Point deſcribing the Curve, obtain in the reſpective Points M of it; as likewiſe the trilineal Figures *a b m*, *a c m*, *a d m*, do exhibit the aggregatal Velocities from the beginning of the Motion to the reſpective definitive Inſtants of Time; to which (as we have long ſince obſerved) the correſpondent Spaces BM, CM, DM are proportional.

18. From what has been ſaid it is evident, if a *Circle*, *Ellipſis*, or any other Curve returning into it ſelf be conceived to be generated after this manner, the Point deſcribing them will have an infinite Velocity in the Point of *Recurſion*, *viz.* if the Quadrant A F M be that generated. Becauſe the Tangent T M is parallel to the Diameter A Z, and will not meet the ſame unleſs at an infinite Diſtance, therefore the Velocity in M will be to the uniform Velocity of the Motion through AY, as an infinite right Line to PM; and ſo the Velocity in M muſt needs be infinite. Thus it is indeed with regard to Curves of this kind; but as to ſuch that when continued run out infinitely, Fig. 25.

like the *Parabola*, and *Hyperbola*, the Velocity of the deſcending Point is finite in any aſſigned Point of the Curve. But laying aſide theſe things, we proceed on to the Explanation of other Properties of our Curve.

Plate I.
page 86.



1 A B C D E Y Y Y Z Z Z Z Z Y
2 A B C D E I X X X X 4 9
3 A A A A A D E E E E E B B B B B C
4 A B F C D B F C B F C B F C B F C
5 A A A A D E E F E Y B B B B C
6 A F C D B F C Z B C B C F
7 A E E C E B E B
8 L K A H G V G H K L B C D E G K L H
9 V A Z Y a B Z Y b C Z Y c D K Z Y d
10 V A Z B Z C Z D Z K Y
11 V A Z B Z C Z D Z
12 V A Z X B Z X C Z X D Z X
13 M A C A C A C A n X Y B B
14 C A B A C m A C M B B
15 A B B B C M A A C A A
16 A B C D
17 A B C Y M A A M X m m D Z Z Z
18 B C Y A A A M M Z Z Z
19 B C D Y A A A A G M P F N H Q K O Z Z Z L
20 Y G K S O D C M E D O G X T A Z P
21 Y F N M X T A P E Q Z
22 S T S S A Y M P
23 T A M P Z B M C N M D Y O M
24 a b c d m n m
25 Y T A M Z A P Z

LECTURE V.

WE are ſtill continuing on our Buſineſs of deducing the Affections of Curves from the afore-propoſed Generations.

1. *The Angles made* by Ordinates and *Tangents* at different Points of the Curve, *are unequal* to one another; thoſe being *leſs that are* neareſt the Point A, or *Vertex.*

For let the right Lines TM, XN be Tangents, and MP, NQ parallel to AY. I ſay the Angle PMT will be leſs than Fig. 26. the Angle QNX.

For continue out the right Line TM; it will meet QN without the Curve, ſuppoſe at E. Alſo XN will cut the Ordinate PM without the Curve, as at H. But becauſe the Points H, N do fall on different ſides the right Line ME; the right Lines ME, NH will interſect each other within the Parallels PH, QE; and ſo the external Angle QNX is greater than the internal Angle QET, that is, than the Angle PMT. W.W.D.

2. Hence we infer by way of *Corollary*, that *Tangents do interſect one another between the Ordinates drawn thro' the Points of Contact*; *as at* F, *between* PM, QN *continued out.*

3. Alſo *the Angle* PTM *is greater than the Angle* QXN, *(viz. the external Angle greater than the internal one.)*

4. Alſo the Ordinates that be neareſt the Vertex (and ſo any right Lines parallel to them,) do fall upon the Curve more oblique than thoſe more remote.

In *Conick Sections* this has been long ſince demonſtrated by *Apollonius*, as may be ſeen in the ſixth Book of his *Conicks* lately publiſhed.

Fig. 26. 5. If the Angle of Application TAY be a right or obtuſe Angle, I ſay *the Arch* MN *of the Curve is greater than the right Line* NH, *but leſs than the right Line* ME.

For joyn the Subtenſe MN and draw the right Line NR parallel to ZA. And becauſe the Angle XPH is not leſs than a right Angle, the external Angle NHP, greater than it, will be obtuſe. Therefore

fore the right Line NM is greater than NH; and ſo the Arch NM being greater ſtill, will be greater than NH.

And becauſe the Angle RNE equal to XQE is not leſs than a right Angle, it ſhall be $RE \sqsubset RN$. Wherefore $MR + RE \sqsubset MR + RN$; that is, $ME \sqsubset MR + RN$. But (by what is aſſum'd by *Archimedes)* $MR + RN \sqsubset$ Arch MN. Therefore ME is ſtill leſs than the Arch MN.

4. This Theorem is very uſeful in expediating *the Demonſtration of Tangents.* For it follows from hence, if the Arch MN be ſuppoſed indefinitely ſmall, we may ſafely uſe inſtead of it either a Particle ME or NH of a Tangent.

I ſhall here by way of Specimen lay down a general Way of determining the Tangents of Cycloids, and Curves deſcribed like them, ſtrengthen'd by a Demonſtration received from hence.

Let a right Line AY move along any Curve APX, Convex or Concave the ſame Way, with a uniform Motion always parallel to it ſelf, *viz.* paſſing over equal Parts of the Curve in equal Times, and at the ſame Time let ſome Point move from A along AY with an equal Motion too: The *Curve* AMZ will be generated by this Point thus moved, to any Fig. 27.

any given Point M of which it is required to draw a Tangent. In order to which, draw the right Line MP parallel to AY, cutting the Curve APX in P; and thro' P draw the right Line PE touching the Curve APX; to which thro' M draw MH parallel; and in this take any Point R, and draw RS, parallel to PM; then make as the Curve AP to the right Line PM (that is, as one uniform Motion to the other) ſo is MR to RS, and join MS; this will touch the Curve AMZ. For in this Curve take any Point Z, thro' which draw the right Line ZX parallel to MP, cutting the Curve APX in X, its Tangent in E, and MR parallel to it in H, and laſtly MS in K.

Firſt, Let the Point Z be above M towards A; whence the right Line PE $\sqsupset$ Arch PX. And ſo $PA : PE \sqsubset$ Arch $PA : PX :: PM : PM - XZ :: PM : EH - XZ :: PM : ZH - EX \sqsubset PM : ZH$. Wherefore (by permutation) it ſhall be $PA : PM \sqsubset PE : ZH$. But it is $PA : PM :: MR : RS :: MH : KH :: PE : HK$. Therefore $PE : HK \sqsubset PE : ZH$. And ſo $HK \sqsupset ZH$. But the Point H is without the Curve AZM; becauſe $EZ \sqsupset XZ \sqsupset PM = EH$. Therefore it is evident the Point K falls without the Curve AZM.

2*dly*, Let the Point Z be below the Point M. Then will the right Line P E be greater than the Arch P X. Whence the Arch $PA : PE \sqsupset$ Arch $PA : PX :: PM : XZ - PM :: PM : XZ - EH :: PM : XE + XZ \sqsupset PM : HZ$. And contrariwiſe, $PA : PM \sqsupset PE : HZ$. But (as at firſt) it is $PA : PM :: PE : HK$. Therefore $PE : HK \sqsupset PE : HZ$; and ſo $HK \sqsubset HZ$. Therefore it is evident the Point K falls without the Curve, conſequently the whole right Line M K Z lies without the Curve, and touches it at M. W.W.D.

This is by the Way. We now return to other Properties of our Curve.

7. If any Parallel E F be drawn to whatſoever Tangent M T; *viz.* from any Point E taken below T; it will meet the Curve.

For if any Point below M be taken in the Curve, and from the ſame a Tangent be conceived to be drawn; it will meet T M the Tangent below the Ordinate P M. Therefore the right Line E F will meet it too; but it muſt firſt croſs the Curve. Wherefore the thing propoſed is manifeſt. Fig. 28.

8. It appears too at the ſame Time, that the right Line E F will cut the Curve twice, both above and below the Point M.

Apollonius has given us operoſe Demonſtrations of theſe two Concluſions, in the 27*th* and 28*th* *Propoſitions* of his firſt Book of *Conicks.*

Moreover in order to a full Determination of the Places of Concurrence, the particular *Mode* or *Law* of the *deſcendent* and *tranſverſe Motions* ought to be known; and then they may be preſently obtained by *Analyſis.*

Fig. 29. 9. If any two right Lines (HM, KN) be equally inclin'd to a given Curve, that is, if they make equal Angles with the Tangents (ſuppoſe MT, NX) at the Points of Contact; theſe will diverge outwardly, or will concur towards EF.

For draw the ſubtenſe NM; which (by what has been already ſaid) will meet AZ, ſuppoſe in O. Therefore the Angle OMH $\sqsupset$ (the Angle TMH $=$ Angle XNK $\sqsupset$) Angle ONK. Therefore the Angle HMN $+$ MNK $\sqsubset$ 2 right Angles. Wherefore the right Lines HM, KN do concur towards EF. This is true only when the equal Angles HMA, KNA do fall the ſame way, or the one external to the other internal.

Fig. 30. *Apoll.* 38, &c. 5. 10. If the right Line HM be perpendicular to a *Curve*, or to its Tangent MT, which is the ſame thing, and in this be taken any definite Part HM;

HM ſhall be the ſhorteſt of all Lines drawn from H to the Curve.

For draw any right Line HO. This will firſt meet the Tangent, ſuppoſe in R. It is manifeſt H R is greater than H M; and H O is ſtill greater than H M.

11. Hence a Circle deſcribed from H thro' M will touch the Curve.

12. And contrariwiſe; if HM be the leaſt of all Lines that can be drawn from H to the Curve; H M ſhall be perpendicular to the Curve.

For becauſe HM is ſuppoſed a *Minimum*, a Circle deſcribed from H, with any Diſtance HC, greater than HM, will cut the Curve; and ſo the Tangent M T, ſuppoſe (this) in R. Therefore ſince HR ⊏ HM, the Angle H R M will not be a right Angle. The ſame Thing is manifeſt of all the Points in the right Line T M. Therefore the Perpendicular to the Tangent can fall no where but in the Point M.

13. Alſo if the right Line H M be the leaſt of all the Lines that can be drawn from H, and the right Line T M be perpendicular to it; this will touch the Curve. Fig. 31.

For (if poſſible) let ſome other Line X M touch it; then will X M be perpendicular to H M: Conſequently the Angles H M X, H M T will be equal;

the Whole and the Part; which is impoſſible.

14. I ſay moreover, the Line HN which is nearer the leaſt HM, is leſs than HO being more remote.

Fig. 32. For draw the Subtenſe MN; this will paſs beyond the Curve if produced, and cut H O, ſuppoſe in R. And becauſe the Angle HMR is obtuſe (*viz.* greater than that which the Tangent at M forms with HM;) HNR ſhall be more obtuſe; and ſo the right Line HR ⊏ HN, and HO ſtill greater than HN.

15. Hence it is plain a Circle deſcribed from the Center H meets the Curve but in one Point on the ſame ſide M; and ſo at moſt can cut it but in two Points.

16. Let the right Lines IN, KO be
Fig. 33. parallel to the perpendicular HM; IN the neareſt of them will cut the Curve leſs oblique than KO more remote.

For thro' N, O draw EN, FO perpendicular to the Curve; theſe will meet HM within the Curve, ſuppoſe at R, P; and meet each other in Q.

It is now manifeſt the Angle FOK = Ang. FPH ⊏ Ang. PRQ = Ang. NRH = Ang. ENI. Therefore ſince the Angle FOK is greater than the Angle ENI, the thing propoſed is manifeſt.

17. If from any Point H taken in the perpendicular H M be drawn the right Lines H N, H O to the Curve; the nearest of theſe (H N) will nearer approach a right Angle, than H O, which is more remote. Fig. 34.

For draw E N, F O perpendicular to the Curve, and I N, K O parallel to H M. Therefore is the Ang. F O K $\sqsubset$ E N I. Alſo the Ang. O H M $\sqsubset$ Ang. N H M. That is, the Ang. K O H $\sqsubset$ Ang. I N H. Wherefore the Ang. F O K + K O H $\sqsubset$ Ang. E N I + I N H. That is, the Ang. F O H $\sqsubset$ Ang. E N H. Therefore the Propoſition is evident.

18. Hence it is manifeſt the *Obliquity* of the Lines falling on the Curve from the Point H, continually increaſes, the farther they fall from the perpendicular, until you come to that which touches the Curve, which is the moſt Oblique of them all.

19. Moreover if the Point H be taken now within the Curve, and H M be the leaſt of all Lines drawn from it to the Curve; H M will be perpendicular to the Curve, or Tangent M T. Fig. 35.

For if any other Line M R be ſaid to be perpendicular to the Tangent, then will be H R $\sqsupset$ H M; and H O ſtill leſs than H M. Conſequently H M is not a *Minimum*, *Apoll.* 32, 5.

Minimum, which is contrary to the *Suppoſition.*

Apoll. 29. 5. 20. Alſo if the right Line HM be the greateſt of all Lines falling from H upon the Curve; the Line HM will be perpendicular to the Curve. Fig. 36.

For a Circle deſcribed from the Centre H thro' M will fall quite without the Curve. Therefore if the right Line MT touches the Circle, this will fall more without the Curve, and accordingly will touch the ſame. But HTM is a right Angle. Therefore the Propoſition is evident.

Apoll. 30, 39. 5. 21. If MT be perpendicular to the *Minimum* or *Maximum* HM; this will touch the Curve.

For if any other Line MX be ſaid to touch it; the Angle XMH will be right, and equal to the Angle TMH. Which is impoſſible.

22. Hence the right Line YM be not perpendicular to the Curve, no *Maximum* or *Minimum* can be taken in the ſame.

Apol 31, 47. 5. For if this be poſſible YM muſt be perpendicular to the Curve. Which is contrary to the *Suppoſition.*

23. If HM be the leaſt of the Lines falling on the Curve; and any Point I be taken in it; IM will be a *Minimum* too. *Apoll.* 30, 5.

For becauſe a Circle deſcribed from the Centre H thro' M cuts the Curve inwardly; a Circle deſcribed from the Center I, will touch it more inwardly. Therefore the Propoſition is evident.

24. Alſo if H M be the greateſt of the Lines falling on the Curve, and any Point I be taken without it; I M will be a *Maximum*.

For becauſe a Circle deſcribed from H thro' M touches the Curve outwardly, for a ſtronger Reaſon will a Circle deſcribed from I thro' M touch the ſame outwardly. Whence the Thing propoſed is manifeſt.

Note, A nearer Determination of theſe *Maxima* and *Minima* depends upon the particular Nature of the Curve.

But I ſhall now withdraw my ſelf from this Speculation; not having a Deſign at preſent of propoſing any more of theſe Kind of Propoſitions. It is ſufficient for our purpoſe to have hitherto exhibited a *Specimen* of a Kind of general Doctrine containing the Properties of Curves, which being doubtleſs very full and perfect, ſeems to afford no ſmall Compendium to ſuch Geometrical Matters, that chiefly relate to the Properties and Affections of Curve Lines. Whether then may not I ſay, it appears almoſt a Fault, and not over and above conſonant to the

Rules of Logick, to branch out into parts, or derive from a narrow Spring, thoſe Things that agree to any whole kind, and flow from the ſame common Principle. In time I ſhall, perhaps, propoſe many other things, and ſuch too as are more *difficult*. But I am inclin'd now to leave off.

LECTURE VI.

IT is now long ſince we have laid down and explained ſome general Properties of Curves, from a certain common Principle of Generation, chiefly ſuch which I obſerve are apply'd by the antient *Geometricians* to the particular Curves that they have handled. Now in my Opinion it will not be amiſs if I add ſeveral other Things more abſtruſe yet not unpleaſant or uſeleſs, demonſtrated with our uſual conciſeneſs, in a *Method* as *Scientifick* as I could, not only aſſerting the Truth of the Concluſions, but opening the Springs from whence it flows. What we advance in the main, partly relates to the *Inveſtigation* and *ready Demonſtration* (withal) of *Tangents*, *without* *the Trouble of Calculation*; (always obſerving to deduce what is moſt perplex'd and leſs manifeſt from what is moſt ſimple and common) and partly *to the ready Determination of the Dimenſions of many Magnitudes by means of their Tangents.* Which ſubjects as they ſeem to have their Difficulties, with regard to other Geometrical Matters, ſo they have not been ſo fully handled or exhauſted, as ſome others, at leaſt not to my Knowledge, after the Manner we have done. In order to which, in the firſt Place we lay down the following Propoſitions as ſo many *Lemma*'s affording ſome Aſſiſtance in making the Demonſtrations more ſhort and eaſy.

Fig. 38. 2. Let ABC be a *right-lin'd Angle*, and D a given Point, alſo let the Line ODO be ſuch, that drawing any right Line DN thro' D; let MN intercepted between the Legs of the Angle, be equal to the Part DO, lying between the Point D and the Line ODO; the Line ODO will be an *Hyperbola*.

For draw DL parallel to CB, meeting AB in the Point L. In BL continu'd take LZ = LB; draw ZS parallel to BC; and OK parallel to BZ. Then becauſe it is put DO = MN; it ſhall be HO = BN; therefore ſince we have DH : HO :: (DL : LN :: DL — DH : LN —

$HO :: LH : LB ::) LH : HK$. It ſhall be $DH \times HK = HO \times LH$; that is, $DL \times HK - LH \times HK = KO \times LH - HK \times LH$. Therefore it ſhall be $DL \times HK = KO \times LH$. Or $ZL \times LD = ZK \times KO$. Wherefore it is manifeſt the Line ODO is an *Hyperbola*, whoſe *Aſymtotes* are ZA, ZS. This may be demonſtrated ſhorter by continuing out the right Line NDS. For it is $DS = DM = DO \pm OM = NM \pm OM = ON$. In like manner may the fourth and ninth Propoſitions be demonſtrated ſhorter.

Fig. 38. Even if MN to DO be ſuppoſed always to have the ſame given Ratio, which let be as the given Quantity R to S; the Line ODO will be ſtill an *Hyperbola*. For making $R : S :: LB : LZ$; and $R : S :: DL : DE$; drawing ZS thro' Z parallel to BC; and RE thro' E parallel to ZA, meeting ZS in the Point Y. This done, YR, YS will be the *Aſymtotes* of the ſaid *Hyperbola* which now is enough to have obſerved.

Hence we may take Notice by the Way, how eaſy it is to reſolve this *Problem (whereby thoſe Solutions of* Archimedes's *and* Vieta's *Problems effected by Means of the firſt Conchoid, are reduced to the Conick Sections)* To draw a right Line thro' a given Point D, ſo that the Part of the Line drawn intercepted between the ſides of a given Angle ABC may be equal to a given Line T. For the *Hyperbola* ODO being deſcribed; from the Centre D, with a Diſtance equal to the given Line T deſcribe a *Circle* POQ interſecting the *Hyperbola* in the Point O; continue out DON, and make $MN = DO = T$. This is a more general and conciſe Method than that we have given in our *Opticks*.

Fig. 39. 4. Let ABC be an Angle, and D a given Point; alſo let the Line OBO be ſuch, that drawing any right Line DN thro' D, the Line MN intercepted by the Legs of the Angle, to MO intercepted by the right Line BC and Curve OBO, be always in the ſame given Ratio (ſuppoſe X to Y): the Line OBO ſhall be an *Hyperbola*.

For draw the right Line DL parallel to CB, meeting AB in L; divide DL, BL in the Points E, F, ſo that $DL : DE :: X : Y :: BL : BF$; thro' E draw the right Line ER parallel to BA, and thro' F the right Line FS parallel to BC; let the right Lines ER, FS meet in the Point Z; and laſtly, thro' the Point O draw OH parallel to AB. Now becauſe $DL : DH :: LN : HO :: LB + BN : HO :: DE \times LB + DE \times BN : DE \times HO$. Alſo $DL \times KO = DE$

$= DE \times BN$ (for $DL : DE :: MN : MO :: BN : KO$) and $DE \times LB = DL \times BF$ (ſince $DE : DL :: BF : LB$) it ſhall be $DL : DH :: DL \times BF + DL \times KO : DE \times HO$; that is, $DL \times BF + DL \times KO : DH \times BF + DH \times KO :: DL \times BF + DL \times KO : DE \times HO$. Therefore $DH \times BF + DH \times KO = DE \times HO$; that is, $DH \times BF + DH \times HO - DH \times BL = DE \times HO$; and therefore (by Tranſpoſition) $DH \times HO - DE \times HO = DH \times BL - DH \times BF$. That is, $EH \times HO = DH \times FL$; or $EH \times GO + EH \times HG = DE \times FL + EH \times FL$. Wherefore, taking away equals, it is $EH \times GO = DE \times FL$; or $ZG \times GO = DE \times FL$. And ſo becauſe $DE \times FL$ is determinate, it is evident the Line OBO is an *Hyperbola*, whoſe *Aſymtotes* are ZR, ZS.

Fig. 40. 5. If MO be taken on the other ſide the right Line BC; DE, BF muſt be taken alſo on the other ſide the Points D, B, as appears by the Scheme. Nor is the manner of Demonſtration different.

Fig. 41. 6. Conſectary. If the right Line BQ cuts the Angle ABC, and thro' the Point D be drawn any how two right Lines MN, XY cutting the right Line BQ in the Points O, P (whereof O is nearer than B) it ſhall be MN : MO ⊐ XY : XP. For thro' O conceive to be deſcriben an *Hyperbola* VOB, (ſuch as that but juſt now mentioned, *viz.* that the intercepted Lines may have the Ratio of MN to MO). Then ſhall MN : MO :: (XY : XV) ⊐ XY : XP.

Coroll. By Diviſion it is NO : MO ⊐ YP : PX.

Fig. 42. 7. Moreover if there be ſeveral Lines BQ, BG which cut the Angle ABC, and from the Point D be drawn the right Lines DN, DY (cutting the other right Lines, as *per* Scheme, of which DN is the neareſt the Point B;) it ſhall be NE : MO ⊐ YF : VX.

For NE : EO ⊐ YF : FV, and EO : OM ⊐ FV : VX. Therefore (by equality) NE : OM ⊐ YF : VX.

8. Hence it is manifeſt too, that a right Line may be drawn (either Way) thro' B ſo, that the Parts of right Lines drawn from D intercepted by it and the right Line BC, may have a Ratio to the Parts intercepted by the right Lines BA, BC, leſs than any aſſignable Ratio.

For take PQ = PZ; then BQ (joyn'd) touches the *Hyperbola* OBO; and it is manifeſt the Parts intercepted by BQ, BC have a leſs Ratio to the Parts intercepted by BC, BA, than the Intercepts by the *Hyperbola* OBO, and the right Line BC to the ſame, that is, leſs than any given Ratio.

9. Let again ABC be a right-lin'd Angle and D a given Point; alſo let the Line OOO be ſuch, that any how drawing the right Line DO from D, cutting the Legs of the Angle in the Points M, N; DM may always have the ſame Ratio to NO (ſuppoſe X to Y). The Line OOO will ſtill be an *Hyperbola*. Fig. 43.

For draw DL parallel to BC, and let DL : DE :: X : Y, and thro' E draw ER parallel to AB, cutting BC in Z. Laſtly, Thro' O draw OH parallel to BA.

Now it is DL : DE :: DM : NO :: LM : GO (becauſe of the ſimilar Triangles DLM, NGO) :: LM × DH : GO × DH. Alſo DL × HO = LM × DH (becauſe DL : LM :: DH : HO). Wherefore DL : DE :: DL × HO : GO × DH. That is, DL × HO : DE × HO :: DL × HO : GO × DH. And ſo DE × HO = GO × DH. And ſo DE × HO = GO × DH. That is, DE × HG + DE × GO = GO × DE + GO × EH. Wherefore (taking away what is common) it is DE × HG = GO × EH; or DE × HG = GO × ZG.

Therefore it is manifeſt the Curve O O O is an *Hyperbola*, whoſe *Aſymtotes* are Z R, Z C.

Coroll. If the given Ratio be one of equality or (D M = N O ;) A B, C B will be then the *Aſymtotes.*

Some of the following Propoſitions we ſhall ſhew Algebraically for Perſpicuities ſake.

Fig. 44. 10. Let I D be a right Line given in Poſition, and D an aſſigned Point ; let alſo the Curve DNN be ſuch that taking any Point G in I D, and drawing a right Line G N parallel to I K given in Poſition; aſſuming the determinate right Lines m, b; putting D G $= x$, and G N $= y$; I ſay when this is done, if it be always $my + xy = \frac{m}{b} xx$; the Line D N N ſhall be an *Hyperbola*, which may thus be determin'd. Aſſume D M, D O (each way) equal to m; thro' m draw M L parallel to I K; make $b : m :: m :$ M Q; let M Z $= 2$ M Q $= \frac{2mm}{b}$; then thro' Z, O draw the right Line Z T ; the Lines Z M, Z T will be the *Aſymtotes.*

For draw Z S parallel to M O, meeting N G in R (which laſt cuts Z T in P,) and joyn D Q. Then is P N = R G + G N — R P. But it is M D : M Q ::

ZR (MG) : RP; that is, $m : \frac{mm}{b} :: m + x : \mathrm{RP} = \frac{mm}{b} + \frac{mx}{b}$; and ſo RG — RP $= \frac{mm}{b} - \frac{mx}{b}$. Therefore PN $= \frac{mm - mx}{b} + y$. Whence PN × MG $= \frac{m^3}{b} + my + xy - \frac{mxx}{b}$. But (by Suppoſition) it is $my + xy - \frac{mxx}{b} = 0$. Therefore PN × MG $= \frac{m^3}{b} =$ MD × ZQ. Or PN : ZQ :: (MD : MG ::) QD : ZP. Wherefore PN × ZP = ZQ × QD. Therefore it is manifeſt the Curve DNN is an *Hyperbola*, whoſe *Aſymtotes* are ZM, ZT.

11. Obſerve; If the Equation be $my - xy = \frac{m}{b}xx$; the ſame *Hyperbola* will be had; the Points G being then taken towards DM. So likewiſe when the Equation is $xy - my = \frac{m}{b}xx$; by taking the Points G beyond M, the Conjugate *Hyperbola* to this will be had.

Fig. 45. 12. Let BDF be a Triangle, and DNN ſuch a Line, that drawing any how RN parallel to BD (cutting the Lines BF, DF, DNN in the Points R, G, N) and joyning the right Line DN; DN may always be a mean Proportional between RN, NG; the Line DNN will be an *Hyperbola*.

Thro' D draw DK perpendicular to DB (cutting RN in E) and let FH be parallel to DB. Call DB, b; DF, d; FH, f; DG, x; and GN, y. Then is $d : f :: x : \frac{fx}{d} =$ GE. Whence $\frac{2fxy}{d} + xx + yy = 2\,EG \times GN + DGq + GNq = DNq$. Again it is $d : b :: FG : GR :: d - x : RG = b - \frac{bx}{d}$. Whence $RN = b - \frac{bx}{d} + y$. And ſo by $-\frac{bxy}{d} + yy = RN \times NG = DNq = \frac{2fxy}{d} + xx + yy$. Wherefore by $-\frac{bxy}{d} = \frac{2fxy}{d} + xx$. And thence $\frac{db}{2f+b}y - yx = \frac{d}{2f+b}xx$. That is, (putting $m = \frac{db}{2f+b}$) $my -$

$xy = \frac{m}{b}xx$. Whence it is manifeſt DNN is an *Hyperbola*, ſuch as has been determined above. *Note*, if the Angle DGN be a right Angle, then $f = o$ vaniſhing, we ſhall have $d = m$, or $dy - xy = \frac{d}{b}xx$. We inſert ſome other things here, among which are perhaps Acquiſitions.

Let the right Line ID be given in Poſition; alſo let the Curve DNN be ſuch, that taking any Point G in ID, drawing the right Line GN parallel to IK given in Poſition, taking the determinate Lines g, m, r, and putting $DG = x$, and $GN = y$, it may be always $yx + gx - my = \frac{m}{r}xx$; the Line DNN will be an *Hyperbola* thus determinable: Aſſume $DM = m$, and thro' M draw ML parallel to IK; and in this take $MQ = \frac{mm}{r}$, and let $QY = MQ$, and from MY cut off $YZ = g$; and joyn QD, and draw ZT parallel to it: then will ZM, ZT be the *Aſymtotes*. Fig. 46.

For draw ZS parallel to MD, meeting GN continued out in the Point R, and let GR cut ZT in P. Now it is $PN =$

$RG - RP - GN = \frac{mm}{r} - g + \frac{mx}{r} - y$. And ſo $PN \times MG = \frac{m^3}{r} - mg + yx + gx - my - \frac{m}{r} xx = \frac{m^3}{r} - mg + o = \frac{m^3}{r} - mg = DM \times ZQ$. Wherefore $PN : ZQ :: (DM : MG ::) QD : ZP$. Therefore $PN \times ZP = ZQ \times QD$. Wherefore it follows the Curve DNN is an *Hyperbola*, whoſe *Aſymtotes* are ZM, ZT.

If the Equation be $- yz + gx + my = \frac{m}{r} xx$; it will be the ſame *Hyperbola*. But then the Points G muſt be taken between B, M; and ſo as the Situation of the Points G vary, the Signs of the Terms of the Equations do ſo too; but this is now not a Place to expound theſe Things.

Fig. 47. 14. Let the right Lines DB, BA be given in Poſition, and let the right Line CX move along the right Line DB parallel to BA; alſo let a right Line DY revolving about the Point D ſo cut BA in E, that there be always the ſame Ratio between BE, DC (ſuppoſe that which ſome Quantity R has to DB) and let the right Lines DE, CX interſect one another

ther in the Points N; the Line D N N will be a *Parabola.*

For let R : D B :: D B : P. Then is B E : D C :: D B : P. Alſo it is D B : B E :: D C : C N. Therefore D B : B E + B E : D C = D C : C N + D B : P. That is, D B : D C :: D C $\times$ D B : C N $\times$ P. That is, D B $\times$ D C : D C*q* :: D C $\times$ D B : C N $\times$ P. Wherefore it is D C*q* = C N $\times$ P. Whence it is manifeſt the *Curve* D N N is a *Parabola*, whoſe *Parameter* is P, *Vertex* D, and *Diameter* parallel to B A.

Gregory St. *Vincent* has given us this Theorem, but * Demonſtrated (if I remember aright) by an operoſe Prolixity.

* In his Book of the *Spiral.*

15. Moreover the ſame Things being ſuppoſed, if C X, D Y be ſo moved, that B E, B C have now always the ſame Ratio (ſuppoſe that of B C to R;) the Interſections will be alſo in a *Parabola.* Fig. 48.

For biſect D B in G, and draw G V parallel to B E, cutting the Curve D N N in V; and becauſe B C : R :: B E : B D :: C N : C D, it ſhall be B C $\times$ C D = R $\times$ C N. Therefore (by a well-known Property of the *Parabola*) the Curve D N N is a *Parabola*, whoſe *Parameter* is R, and *Diameter* G V

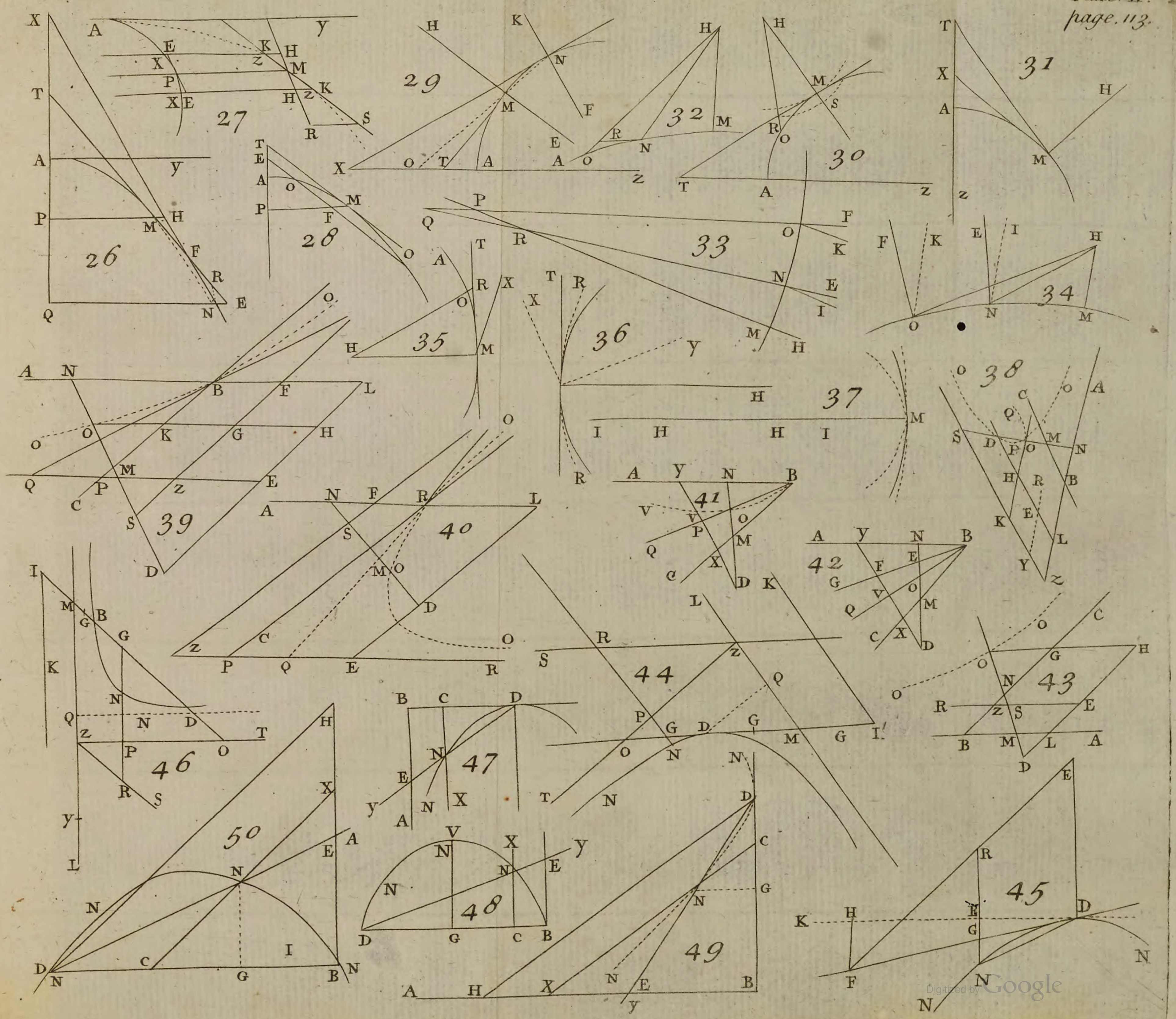
Theſe are perhaps common: But what follows cannot be ſaid to be ſuch.

Fig. 49. 16. Things remaining as before, let the right Line CX now, inſtead of moving parallel to BA, move parallel to ſome other Line DH given in Poſition; and let it always be BE : DC :: DB : R: the Interſections N will be in an *Hyperbola.*

For draw NG parallel to BA; call DB, b; BH, h; DG, x; GN, y. Then is $x : y :: b : \frac{by}{x} =$ BE. Alſo $b : h :: y : \frac{by}{h} =$ GC. Wherefore CD $= x - \frac{by}{h}$. Therefore (by the Suppoſition) $\frac{by}{x} : x - \frac{by}{h} :: b : r$; and ſo $yx + \frac{hry}{b} = \frac{h}{b}xx$. And (putting $\frac{hr}{b} = m$) $yx + my = \frac{m}{r}xx$. Therefore the Curve DNN is an *Hyperbola*, which we have determined in the 10*th* Number of this Lecture.

17. Moreover (things remaining the ſame as in our laſt) if CX be now moved ſo, that BE has always the ſame Ratio to BC, as BD to R; the Interſections N will ſtill be in an *Hyperbola.*

For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For draw NG paralſel to AB, and name the right Lines as before. Then it is BC $= b - x + \frac{by}{b}$; and $\frac{by}{x} : b - x + \frac{by}{b} :: b : r$. Whence we get the following Equation $yx + bx - \frac{br}{b}y = \frac{b}{b} \times xx$; that is, (putting $\frac{br}{b} = m$) yx Fig. 50. 51. 52.
$+ bx - my = \frac{m}{r}xx$. Therefore the Curve BNN is an *Hyperbola*, being ſuch as is above determin'd.

18. Let the right Lines DB, BA be given in Poſition, and D an aſſigned Point in DB, and let the Line DNN be ſuch, that drawing GN any how parallel to BA; aſſuming the determinate Lines g, r, calling DG, x; and GN, y; it may be $ry - yx = gx$; the Line DNN will be an *Hyperbola*, thus to be determin'd. Fig. 35.

Take DE $= r$, and BO $= g$; and thro' E draw the right Line ER parallel to BA, and thro' O the right Line OS parallel to BD: Then will ZR, ZS be the *Aſymtotes*.

For if NP be drawn parallel to DB, it is ZP $= g + y$, and PN $= r - x$; Wherefore ZP $\times$ PN $= gr - gx + ry - yx$. But (by Suppoſition) $-gx$

$+ ry - y\dot{x} = 0$. Therefore Z P × P N $= gr =$ Z E × E D. Whence the thing propoſed is manifeſt.

If the Equation had been $xy - ry = gx$; D E muſt be taken equal to r, and B O $= g$ (below the Line D B). And drawing the Parallels S Z, Z R, as before; the Line N N N contained within the Angle S Z R will be an *Hyperbola*, as may eaſily be prov'd after the ſame way.

19. Let D B, B A be right Lines given in Poſition; and let the right Lines F X be ſo moved parallel to D B, and D Y paſſing thro' a given Point D, that the Ratio of B E to B F be always equal to a given Ratio, *viz.* that of D B to R: Then will the Interſections of the right Lines D Y, F X be in a *right Line.*

For thro' N draw G K parallel to B A; then is D B : D G :: B E : G N :: B E : B F :: B D : R. Wherefore it is always D G = R. And ſo by making D G = R, and drawing G K parallel to B A, it is manifeſt all the Interſections will lie in this Line.

20. The ſame Things being ſuppoſed, and ſome other Point O being taken in B A, and the Computation is to begin from this, *viz.* that it be perpetually B E : O F :: D B : R; the Interſections N will be in an *Hyperbola.*

For draw N G parallel to AB, let D B be $= b$; OB $= g$; D G $= x$; GN $= y$. Then is BE $= \frac{by}{x}$, and O F $= g + y$. Conſequently $\frac{by}{x} : g + y :: b : r$. And ſo $ry - yx = gx$. Wherefore D N N is an *Hyperbola* juſt now determined above.

If the Point O be taken below DB; the Equation will be $yx - ry = gx$. Whence the thing is again evident.

21. The ſame Things being ſuppoſed, let the right Line FX now, inſtead of moving parallel to DB, move parallel to ſome other Line D H, ſo that taking a Point in B A; BE may have always an aſſign'd Ratio to O F (ſuppoſe that of D B to *m*;) the Interſections N will be ſtill in an *Hyperbola*. Fig. 54.

For draw NG parallel to A B; call D B, b; H B, f; H O, g; D G, x; G N, y. Then is $x : y :: b : \frac{by}{x} =$ B E; and $b : f :: x : \frac{fx}{b} =$ G K. Wherefore NK (F H) $= y + \frac{fx}{b}$, and O F $= y + \frac{fx}{b} - g$. Therefore it is $\frac{by}{x} : y + \frac{fx}{b} - g :: b : m$. Conſequently we get this Equa-

tion

tion $my + gx - yx = \frac{f}{b}xx$. Or making $f : b :: m : r$, it is $my + gx - yx = \frac{m}{r}xx$. Wherefore it is evident the Line DNN is an *Hyperbola*; ſuch as is determined above.

Obſerve, If the Computation takes it's beginning from the Point H (that is, if $BE : HF :: DB : m$) the Term g then vaniſhing, we ſhall have $my - yx = \frac{m}{r}xx$. And therefore there is too another more ſimple Determination above.

Fig. 55. 22. Let ADB be a Triangle, and DYY ſuch a Line, that any how drawing PM parallel to DB; it may be $PY = \sqrt{\overline{PMq - DBq}}$: the Line DYY will be an *Hyperbola*, whoſe *Centre* is A, *Semi-diameter* AD (or *Aſymtote* AB) and *Semi-parameter* P; by making $AD : DB :: DB : P$.

* 6. 2. *Elem.* For let it be $TD = 2\,AD$. And it is $AD : P :: (ADq : DBq :: APq : PMq ::$ * $TP \times DP + ADq : PMq :: TP \times DP : PMq - DBq ::) TP \times DP : PYq$. Or $TD : 2P :: TP \times DP : PYq$.

Whence the Propoſition is manifeſt.

Coroll. If YS touches the *Hyperbola* DYY; it ſhall be $PMq : PYq :: PA : PS$.

For it is $PMq : DBq :: PAq : ADq :: PA : AS$. And ſo (inverſely) it is $PMq : PYq :: PA : PS$.

23. The ſame Things remaining, let now PY be $= \surd PMq + DBq$: the Line YYY will alſo be an *Hyperbola*; A being the *Centre*; AF the *Semi-diameter* (parallel and equal to DB,) and P the *Parameter*, by making $AF : AD :: AD : P$. Fig. 56.

For draw YK parallel to AP meeting AF in the Point K; and let FT be $= 2\,FA$; and it is $AF : P :: (AFq : ADq :: DBq : ADq :: PMq : APq :: PYq - DBq : APq :: AKq - AFq : KYq ::)\ TK \times FK : KYq :: AF : P$. Whence the thing propoſed is evident.

Coroll. If the right Line YS touches the *Hyperbola*, FYY it ſhall be $PMq : PYq :: PA : PS$.

For AD is a *Semidiameter* conjugate to AF. Whence $PA : AS :: PAq : ADq :: PMq : DBq$. Therefore $PA : PS :: PMq : PMq + DBq :: PMq : PYq$.

24. Let ADB be a Triangle, having the right Angle ADB, and the Curve CGD ſuch, that drawing any right Line FEG parallel to DB (cutting the ſaid Fig. 57

Lines as *per* Scheme) the Aggregate of the Squares of EF, EG may be equal to the Square of DB: the Curve CGD will be an *Ellipſis*, and AD, AC the *Semi-Axis*.

For let AV = AD. And it is AD*q* : DB*q* :: (AC*q*) AE*q* : EF*q* : : AD*q* — AE*q* : DB*q* — EF*q*. That is, AD*q* : AC*q* :: VE × ED : EG*q*. Whence the thing propoſed is maniſeſt.

Obſerve, If GT touches the *Ellipſis* CGD; it is EF*q* : EG*q* : : EA : ET.

For becauſe AE : AD : : AD : AT, it is AE*q* : AD*q* :: AE : AT. Wherefore AE*q* : AD*q* — AE*q* :: AE : AT — AE. That is, EF*q* : DB*q* — EF*q* :: AE : ET. That is, EF*q* : EG*q* :: AE : ET.

Fig. 58. 25. Let DTH be a right-lin'd Angle, and A a given Point in the Side DT. Alſo let the Curve VGG be ſuch, that drawing any right Line EFG perpendicular to TD (cutting the Lines TD, TH, VGG, in the Points E, F, G) and joyning AF, it may be EG = AF: the Line VGG ſhall be an *Hyperbola*.

For draw AP, and VPC perpendicular to TH, TD, and PO parallel to TE. Now it is EF*q* = EO*q* (CP*q*) + OF*q* + 2EO × OF (+ 2CP × OF). But becauſe CP : CA :: OP : OF : : CE : OF; it is CP × OF = CA × CE.

Therefore $EFq = CPq + OFq + 2CA \times CE$. Alſo it is $AEq = CEq + CAq - 2CA \times CE$; and ſo it is $EFq + AEq = CPq + CAq + OFq + CEq$ or $EGq = (APq + PFq) = CVq + PFq$. Or $EGq - PFq = CVq$. But it is CE (PO) : PF :: CP : AP :: CP : CV. Whence

$$EGq - \frac{CVq}{CPq} CEq = CVq;$$

and ſo the Line GVG is an *Hyperbola*, whoſe *Centre* is C, and *Semi-Axes* CV, CP.

Note, If the right Line FQ be drawn perpendicular to TH, and you take $QR = AE$; and joyn GR; the Line GR will be perpendicular to the *Hyperbola* VGG. Either rely on me for the Truth of this, or calculate the thing your ſelf; for I ſhall ſpend no Words about it.

Fig. 59. 26. Let AC, BD be right Lines given in Poſition, and AB interſect them; then drawing any how the right Line PKL parallel to AB, (cutting the right Lines AC, BD in the Points P, K) let PL be equal to BK: the *Line* ALL will be *ſtrait*.

For drawing XQ parallel to B A, it is AQ : AP :: (BX : BK ::) QX : PL. Therefore ALL is a *right Line*.

Fig. 60. 27. Let A X be a right Line given in Poſition, and D a Point; alſo let the Line D N N be ſuch, that drawing any right Line M N thro' D (interſecting A X in M, and the Line D N N in N) the Rectangle under D M, D N may be always equal to the Square of a given Line Z: the Line D N N will be the *Circumference* of a *Circle.*

For draw D B perpendicular to A X; and let D B : Z :: Z : D E; joyn N E. then is D M × D N = Z*q* = D B × D E; wherefore D M : D B :: D E : D N. Conſequently the Triangles D B M, D N E are ſimilar; and ſo D N E is a right Angle. Therefore the Line D N N is the *Circumference* of a *Circle*, whoſe *Diameter* is D E.

Hence you may obſerve that not only a *right Line* and *Hyperbola*, but a right Line and Circle too, are (in ſome ſort) Lines reciprocal to one another. But we ſhall reſt a while here, tho' we have not yet done with what we propoſe by way of Introduction.

LECTURE

LECTURE VII.

WE are ſtill remaining at the *Door*; not deſigning as yet to make our Entrance.

1. Let there be two Quantities A, B; whereof A is the greater: If any third Quantity X be taken; it ſhall be $A + X : B + X \sqsupset A : B$.

For becauſe $X : A \sqsupset X : B$; it ſhall be (by Compoſition) $X + A : A \sqsupset X + B : B$. Or (*altern.*) $X + A : X + B \sqsupset A : B$.

2. Let three Points L, M, N be taken in the Line Y Z; and between L, N aſſume any Point E, and without L N towards Z another Point G; divide E G in F, ſo that it be G E : E F :: N L : L M, the Point F will fall towards M Z. Fig. 61.

For

(a) 1. of this. For it is *(a)* NE : ME ⊏ NL : ML :: GE : FE ⊏ NE : PE. Therefore FE ⊏ ME.

Fig. 62. 3. Let two right Lines BA, DC be parallel; as alſo the Lines BD, GP parallel, and thro' the Point B, any how draw two right Lines BT, BS cutting GP in LK, I ſay it will be DS : DT :: KG : LG.

For it is KG : LG = KG : GB + GB : LG = PK : PS + PT : PL = DB : DS + DT : DB = DT : DS.

Fig. 63. 4. Let BDT be a Triangle, and any two right Lines BS, BR drawn thro' B interſect any Parallel PG to the Baſe DB, in the Points L, K: I ſay it will be LG × TD + KL × RD : KG × TD :: RD : SD.

For take BM = GP, and BN = LP, and BO = KP; then it is manifeſt, that PM, PN, PO joyn'd will be (reſpectively) parallel to TB, SB, RB. And becauſe it is DM : PD :: DB : TD, it ſhall be DM × TD = PD × DB. In like manner it is, DN × SD = PD × DB. Wherefore DM × TD = DN × SD = DM × SD + MD × SD. And by Tranſpoſition DM × TD — DM × SD = MN × SD. By the very ſame Way of Reaſoning it is DM × TD — DM × RD = MO × RD. Wherefore it ſhall be MN × SD : MO ×

$RD :: TD - SD : TD - RD$. That is, $LG \times SD : KG \times RD :: TD - SD : TD - RD$, or (bringing the Equation into an Analogy) $LG \times SD \times TD - LG \times SD \times RD = KG \times RD \times TD - KG \times RD \times SD$; and by Tranſpoſition $LG \times SD \times TD + KG \times RD \times SD - LG \times SD \times RD = KG \times RD \times TD$. That is, $LG \times SD \times TD + KL \times SD \times RD = KG \times RD \times TD$. And (reducing this to an Analogy) $LG \times TD + KL \times RD : KG \times TD :: RD : SD$.

5. If the Points T, R be not ſituate on the ſame ſide of D. It ſhall be $LG \times RD - KL \times TD :: RD : SD$. Fig. 64.

The Demonſtration is the ſame, which it would be tireſome to repeat.

6. Let there be four Rows of equal Numbers of Quantities continually proportional, as you ſee below; and let the firſt Antecedents, and the laſt Conſequents be proportional too, *viz.* $A : a :: M : m$ and $E : F :: S : s$. I ſay any four of them taken of the ſame Order will be proportional. Suppoſe for Inſtance $D : d :: P : p$.

A. B. C. D. E. F.
a. b. c. d. e. f.
M. N. O. P. Q. R.
m. n. o. p. q. r.

For Am, Bn, Co, Dp, Eq, Fr.
And aM, bN, cO, dP, eQ, fR.
are continual Proportionals.

Therefore ſince $Am = aM$, and $Fr = fR$, it is manifeſt that $Dp = rP$; and ſo $D : d :: P : p$.

The ſame Concluſion takes place when the Quantities are arithmetical Progreſſionals.

Fig. 65. 7. Let the right Lines A B, C D be parallel, and BD given in Poſition cut them; and let the Lines E B E, F B F be ſo related, that any how drawing P G parallel to D B; P F may be always a mean proportional of the ſame Order between P G, P F; then thro' any given Point E in the Line E B E, let the Line H E be drawn parallel to A B, C D, and let there be another Curve K E K ſuch, that drawing any how Q L parallel to DB, let Q X be always a Mean between Q L, Q I (*viz.* of the ſame Order that P F, is between P G, P E): I ſay the Lines F B F, K E K are Analogous; that is, the Ordinates (as Q R, QR, QK) are perpetually in the ſame Ratio (for Inſtance) that of PF to PE.

This follows from the *Lemma* but juſt laid down, as will appear by the following Scheme.

$$\left.\begin{array}{lll} QS^*, & QR^*, & QI. \\ QL^*, & QK^*, & QI. \\ PG^*, & PF^*, & PE. \\ PE^*, & PE^*, & PE. \end{array}\right\} \begin{array}{l} \text{are -::.} \\ \text{Wherefore} \\ QR : QK :: \\ PF : PE. \end{array}$$

Note, Any parallel Curves, may be uſed inſtead of the right Lines AB, HE, CD.

8. Let again two right Lines AB, AD meet in A, and the right Line BD be given in Poſition; alſo let the two Curves EBE, FBF be ſo related, that any how drawing PG parallel to DB; PF may be always a mean Proportional of the ſame Order between PG, PE; then joyning AE, let there be another Curve KEK ſuch, that drawing any right Line QLI parallel to DB, let QK be always an arithmetical Mean of the ſame Order between QL, QI, as PF was between PG, PE. The Line FEF will again be analogous to KBK; or it will be perpetually QR : QK :: PF : PE. Fig. 66.

For,

$$\left.\begin{array}{lll} QS^*, & QR^*, & QI. \\ QL^*, & QK^*, & QI. \\ PG^*, & PF^*, & PE. \\ PE^*, & PE^*, & PE. \end{array}\right\} \text{are ∺.}$$

$$\left.\begin{array}{ll} \text{Alſo} & QS:QL:: \\ & PG:PE \\ \text{And} & QI:QI:: \\ & PE:PE \end{array}\right\} \begin{array}{l} \text{Therefore } QR: \\ QK::PE:PE. \end{array}$$

Note, Any three analogous Lines may be ſubſtituted for the right Lines AB, AH, AD.

Fig. 67. 11. Alſo, let AGB be a Circle, and D the Centre; and let two other Curves EBE, FBF be ſuch, that drawing any right Line DG thro' D; DF may be always a mean Proportional of the ſame Order between DG, DE; and about the Centre D deſcribe a Circle HE thro' E; then let the Curve KEK be ſuch that drawing any right Line DL thro' D to the Circle HE; DK may be always a Mean of the ſame Order between DL, DI, as DF was between DG, DE; the Curves FBF, KBK will be analogous, or it will be perpetually DR : DK :: DF : DE.

For again,

$$\left.\begin{array}{lll} DS^*, & DR^*, & DI^*. \\ DL^*, & DK^*, & DI^*. \\ DG^*, & DF^*, & DE^*. \\ DE^*, & DE^*, & DE^*. \end{array}\right\} \text{are } ∺.$$

Wherefore D R : D K : : D F : D E.

Any other parallel or analogous Lines may be uſed inſtead of Circles.

10. Let A G B G, E B E be any two Lines; and let F B F be ſome other Line ſo related to theſe, that drawing from a given Point D any right Line D G; D F may be always a mean Proportional of the ſame Order between DG, DE; and take the Line HEL analogous to the Line A G B (or ſuch that any how drawing thro' D the right Line D L S; D S, D L may have conſtantly the ſame Ratio.) Laſtly, Let the Line KE K be ſuch, that any how drawing D L; D K may be always a Mean of the ſame Order between D L, D I, that D F before was between D G, D E; the Line F B F will be analogous to K E K.

For again,

$$\left.\begin{array}{lll} D\,S, & *D\,R, & *D\,I. \\ D\,L, & *D\,K, & *D\,I. \\ D\,G, & *D\,F, & *D\,E. \\ D\,E, & *D\,E, & *D\,E. \end{array}\right\} \text{are} :: .$$

And as well the firſt as four laſt Terms are proportional. Whence the thing propoſed is evident.

11. Let A, B, C, D, E, F be a Series of arithmetical Progreſſionals; in which aſſuming any two Terms D, F; let N be the Number of Terms (excluſively) from A to D, and M the Number (excluſively) from A to F; then ſhall A —: D : A —: F :: N : M.

For let X be the common Difference. Then is D = A ± N X. And F = A ± M X. Wherefore A —: D = N X. And A —: F = M X. Therefore A — : D : A — : F :: (N X :: M X ::) N : M.

12. Hence if there be two Series of this kind, and in each be aſſum'd two Correſpondent Terms in the ſame Order, as D, F in the firſt, and P, R in the ſecond, it ſhall be A — : D : A — : F :: M — : P : M — : R.

A. B. C. D. E. F.
M. N. O. P. Q. R.

For each of theſe Ratio's is equal to that which the Numbers N, M are put to have in our laſt.

And theſe Numbers N, M are call'd the *Indices*, or Exponents of the Terms to which they belong, in any Series of Proportionals; ſuch we always hereafter underſtand, when we expreſs theſe Letters.

13. Let there be any Quantities in a continu'd arithmetical Progreſſion A, B, C, D, E, F, and the ſame Number of geometrical continual Proportionals, beginning from the ſame Term A; and let B the ſecond Term of the former Series, be not greater than the ſecond M of this;

A. B. C. D. E. F.
A. M. N. O. P. Q.

I ſay each Term in the geometrical Series will be greater than the correſpondent Term of the arithmetical Series.

For it is $A + N \sqsubset 2M$ (or $\sqsubset$) $2B = A + C$. Therefore $N \sqsubset C$. Whence $M + N \sqsubset B + C = A + D$. But $A + O \sqsubset M + N$. Therefore $A + O \sqsubset A + D$. And ſo $O \sqsubset D$. Therefore $M + O \sqsubset B + D = A + E$. But $A + P \sqsubset M + O$. Therefore $A + P \sqsubset A + E$;

and ſo $P \sqsubset E$. And you may reaſon on in the ſame Manner as long as you pleaſe.

14. Hence again, if A, B, C, D, E, F be continual arithmetical Progreſſionals, and A, M, N, O, P, Q be continued geometrical ones; and if the laſt F be not leſs than the laſt Q; B ſhall be greater than M.

For if B be ſaid not to be greater than M; F will then be leſs than Q. Which is contrary to the Suppoſition.

Alſo the ſame Suppoſition remaining, the laſt Term but one E, will be greater than the laſt Term but one P.

15. For if $F = Q$, it is evident from our laſt, that $E \sqsubset P$ (*viz.* by inverting each Series;) but if $F \sqsubset Q$, we ſhall have a ſtronger Reaſon to conclude $E \sqsubset P$.

16. Moreover, every Term in the arithmetical Series, is greater than the correſpondent Term in the geometrical Series, as ſuppoſe C greater than N.

For it is $E \sqsubset P$, and ſo $D \sqsubset O$: Whence $C \sqsubset N$.

Fig. 68. 69. 17. It follows from hence, if there be four Lines HBH, GBG, FBF, EBE interſecting each other in B, and ſo related to one another, that drawing a right Line DH any how parallel to DB given in Poſition (*viz.* terminating in the Line DDD) or drawing DH from a given Point D; DG

D G may be always an arithmetical Mean of the ſame Order between D H, D E, with D F a geometrical Mean between them; the Lines G B G, F B F will mutually join one another.

For it is evident from our laſt that the right Line G B H falls quite without the Line F B F.

18. From hence alſo (obſerve briefly by the way) the *Aſymtotes* of innumerable *Hyperbola*'s, or *Hyperboliform* Figures may be determined thus: For let the right Lines V D, B D be given in Poſition; alſo let A B, V I be two other right Lines; and drawing the right Line P G at pleaſure parallel to D B, let P p be always an arithmetical Mean between P G, P E of the ſame Order with P F a geometrical Mean between them. Then becauſe (a) the right Lines E G, E*p*, are always in the ſame Ratio; the Line *p p p* is a right Line: But the Line V F F is an *Hyperbola*, or ſome one of the *Hyperboliform* Figures. (*viz.* the common or *Appollonian Hyperbola*, when P F is a ſimple Mean between P G, P E; but ſome *Hyperboliform* Figure, when P E is a Mean of another Order.) But it is manifeſt (by N°. 16.) the Line *p p p* is an *Aſymtote* correſpondent in the ſame Order to the Line V F F. I know not if it be worth while to obſerve this

Fig. 70.

(a) 12. of this.

here:

here: However it may certainly be eſteem'd an Acquiſition gain'd in our paſſage.

19. Let three right Lines B A, B C, B Q
Fig 71. be drawn from a given Point B to the right Line A C given in Poſition; then in Q C continu'd out aſſume any Point D; a right Line (ſuppoſe B R) may be drawn (on either ſide B Q) ſuch, that any right Line, as D N, being drawn from D, the Part EF of it intercepted between the right Lines BQ, BR, can be leſs than the Part NM of the ſame intercepted between B A, B C.

For firſt, if B R falls beyond the Angle A B C with reſpect to the Point D; make Q R = C A, and joyn B R; then draw D E any how cutting theſe Lines as *per* Scheme. And it is manifeſt * it
* 7. Lect. 6. will be F E ⊐ N M.

If B Q falls on this ſide the Angle
Fig. 72. (*a*) 8. Lect. 6. A B C towards D; (*a*) draw the right Line B H ſuch, that the Intercepts B Q, B H be leſs than the Intercepts B Q, B A, and aſſume H R = Q C, and joyn B R; alſo draw D N any how interſecting the Lines, as *per* Scheme. Now becauſe it
(*b*) *Conſtr.* is K F (*b*) ⊐ N F, and K E * ⊏ M F, it is plain there remains F E ⊐ N M.

So alſo a right Line B R may be drawn on one ſide B Q, which can intercept leſſer Lines than M N; but (*a*) a right Line may be drawn too on the other ſide, intercepting leſs Lines than F E. Whence the whole Thing propoſed is evident.

20. Let D, E, F be three Points in the right Line D Z, and let F be the Vertex of the right-lin'd Angle B F C, whoſe Sides let the right Line D B C cut, and thro' E draw E G: From E we may draw a right Line, as E H, ſuch, that any right Line D K drawn from D, the Part in it intercepted by the right Lines E G, E H may be leſs than the Part intercepted by the right Lines F C, F B. Fig. 73.

Draw E S, E R parallel to F C, F B; and in the firſt Caſe where the Point F is nearer to the Point D, (becauſe of the Similarity of the Triangles E N M, F K I) it is manifeſt that it ſhall be M N ⊐ I K; but a right Line, as E H, may be drawn *(a)* ſuch, that the intercepted Lines P O may be leſs than the intercepted Lines M N. Therefore the Thing is manifeſt. *(a)* 19. of this.

In the other Caſe, where the Point F is neareſt the Point D, take S L equal to C B, and joyn E L. Then is I K : M N :: F K : E N :: D F : D E :: F C : E S :: B C : R S *(c)* :: L S : R S *(d)* ⊏ Q N : M N. Wherefore it is I K ⊏ Q N. *(a)* But a right Line as E H, may be drawn from E ſuch, that the Parts O P intercepted by E G, E H may be leſs than the intercepted Parts Q N. Wherefore the thing propoſed is ſufficiently manifeſt. Fig. 74. *(c)* *Conſtr.* *(d)* 6 Lect. 6.

Fig. 75. 21. Let the right Line BO touch the Curve BA in B, and let the right Line BO be equal to the Curve BA; then taking any Point K in the Curve, joyn the Points K, O; the Line KO will be greater than the Arch KA.

For becauſe a right Line is the ſhorteſt Diſtance between two Points, it is $BK + KO \sqsubset BO = BK + KA$. Therefore $KA \sqsubset KO$.

22. Hence, any how taking two Points K, L on the ſame ſide the Point of Contact, and joyning the right Line KL; it ſhall be $KL + LO \sqsubset KA$.

For, above the Contact towards A, it is $KL + LO \sqsubset KO \sqsubset KA$. But below the ſame, it is $KL + LB \sqsubset KB$ (according to the Hypotheſis of *Archimedes)*; and ſo $KL + LO \sqsubset KA$.

LECTURE VIII.

I Doubtleſs appear to my ſelf (and believe ſhall ſeem alſo to you) to be *that wiſe Scoffer who laughed at great Gates which were to be ſet up, in a very ſmall City:* For all our Endeavours hitherto have been only to get ſomething nearer our Buſineſs; which we are now entring upon.

1. We aſſume theſe Propoſitions in the Nature of Axioms. If two Lines OMO, TMT do mutually touch one another, they comprehend Angles (OMT) leſs than any right-lin'd Angle. And contrariwiſe; if two Lines OMO, TMT do contain Angles leſs than any right-lin'd Fig 76. 77.

Angle, theſe will touch one another; at leaſt may be eſteem'd as Tangents.

The Reaſon of this, if I miſtake not, has been already ſhewn.

2. Hence, if two Lines OMO, TMT, do touch a third Line PMP, theſe Lines OMO, TMT will alſo touch one another.

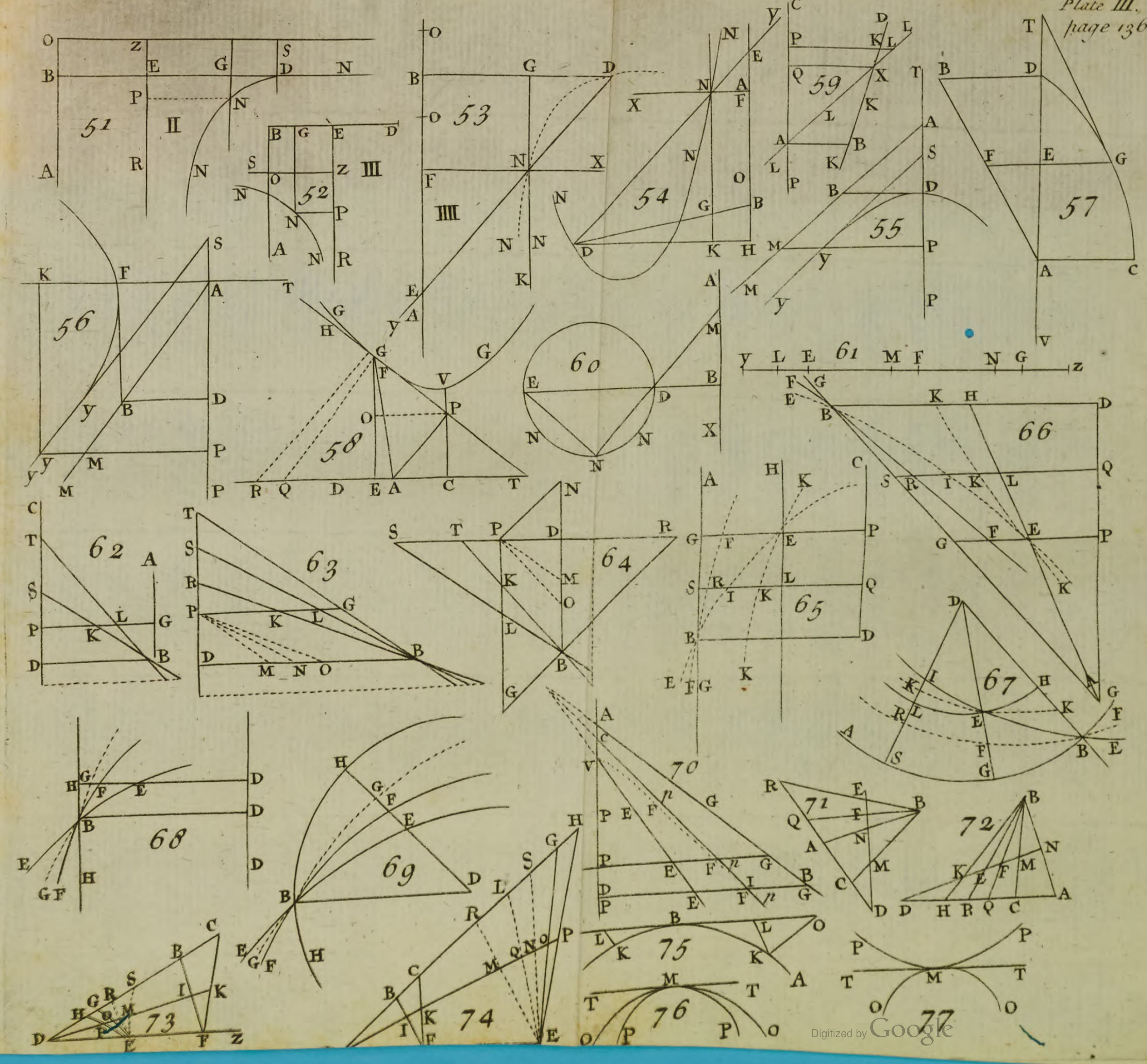
For becauſe the Lines OMO, PMP do touch one another; the Angle OMP will be greater than any right-lin'd Angle. Alſo ſince the Lines TMT, PMP do touch one another, the Angle TMP will be greater than any right-lin'd Angle. Therefore the Angle TMO will be greater than any right-lin'd Angle. And ſo the Lines OMO, TMT will touch one another mutually.

3. Let the right Line FA touch the Curve FX in F, and let the right Line FE be given in Poſition; alſo let the two Curves EY, EZ be ſuch, that drawing any how the right Line IL parallel to EF (cutting the given Lines as *per* Scheme) it may be always, the intercepted Part KL equal to the intercepted Part IG: the Curves EY, EZ will alſo touch one another. Fig. 78.

If they do not touch, a right-lin'd Angle, as BEC, may be conſtituted between them; which is cut by the right Line IL drawn any how parallel to FE; take

GH

Plate III.
page 136.


51 II O Z S B E G D N P N A R N
52 III B G E D S O Z N P N A N R
53 IIII O B G D O N X F N N K E A Y G
54 Y N E X N A F N O N G B D K H
59 C D L P K L Q X A K L B L K P
55 T A S B D M P Y M Y P
57 T B D F E G A C V
56 S K F A T Y B D Y M P Y M P
58 G H G F V P O G R Q D E A C T
60 E D B N N N A M X
61 Y L E M F N G Z
66 F G E B K H D Q S R I K L G F E P K
62 C T A S L G P K D B
63 T S R G P K L B D M N O
64 S T P D R N K M O L B G
65 A H K C G F E P S R I K L Q B D E F G K
67 D I K H K R L E K S F A G B E G F
68 H G F E D B D H D E G F
69 H G F E B D E G F H
70 A C V P E F n G P E F n G I B D E F n G P
71 R E F B Q A N M C D
72 B N K E F M D H R Q C A
73 C B S G R I K H O M D P E F Z
74 H G S L R Q N O P M C B K I F E
75 B O L L K K A
76 M T T O P P O
77 P P T M T O O
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G H = B C, and joyn F H; therefore the Parts of the Parallels to F E intercepted by the right Lines F G, F H will be equal to the Parts intercepted by E B, E C, that is, leſs than the Parts intercepted by the Curve F X and the right Line F A; and ſo the Angle F X A is greater than the right-lin'd Angle H F G; wherefore the right Line F A does not touch the Curve F X. Which is contrary to the Suppoſition.

4. Moreover, let F A be a Tangent to the Curve F X, and let two Curves E Y, Fig. 79.
E Z be ſuch, that any how drawing the right Line D L from a given Point D, (cutting the ſaid Lines as *per* Scheme; it may be always K L = I G; then the Curves E Y, E Z will touch one another.

For if you deny it, conſtitute the right-lin'd Angle B E C between them, which the right Line D L drawn any how from D, interſects. Now we may draw *(a)* (a) 20.
from F a right Line (as F H) ſuch, that Lect. 7.
the Parts of the right Lines drawn from D, intercepted by F G, F H may be leſs than the Parts intercepted by E B, E C; that is, much leſs than the Parts intercepted by the right Line E A and the Curve F X. Whence it follows, the Angle A F X is greater than the right-lin'd Angle G F H; and ſo the right Line A F does not

not touch the Curve FX; which is contrary to the Suppoſition.

The two laſt Concluſions are true, and demonſtrated after the ſame ſort, when the Intercepts IG, KL, have a conſtant given Ratio to each other. I am ſparing of Words.

We have propoſed theſe, to free ſome things following from Doubt.

5. Let VEI be a right Line, and let two Curves YFN, ZGO be ſo related, that drawing the right Line EFG, parallel to AB given in Poſition, the Abſciſs's (a) Hyp. EG, EF may be in the ſame given Ratio; and let the right Line TG touch one of the Curves ZGO in G (meeting the right Line VE in T) the right Line TF being drawn, will alſo touch the other YFN.

For draw IL any how (interſecting the Lines, as *per* Figure). Then IL : IN (a) Hyp. *(a)* $\sqsubset$ IO : IN :: EG : EF :: IL : IK, Therefore IN $\sqsubset$ IK. Wherefore the Point K falls without the Curve YFN, and the right Line TF quite without the ſame.

Otherwiſe, it is IL : IK :: (OI : IN :: IK — OI : IK — IN ::) OL : NK. Therefore ſince the Lines GL, GO do (b) Supp. (c) Schol. 4 of this. *(b)* touch one another, alſo *(c)* the Lines FN, FK will touch one another.

6. Moreover, if there be three Curves XEM, YFN, ZGO ſo related to one another, that any how drawing a right Line EFG parallel to a right Line given in Poſition; EG, EF may be always in the ſame Ratio; and let the Tangents ET, GT of two of them XEM, ZGO meet in the Point T; the right Line TF joyn'd will touch the Curve YFN. Fig. 80.

For (making as before) it will be IL : IK : : EG : EF : : MO : MN. Wherefore (d) the Point K falls without the Curve YFN. This may be otherwiſe Demonſtrated, *viz.* as our laſt, but with more Words. I conceive the Curves to be ſituate, as appears *per* Scheme. For to avoid Verbroſity and diſagreeable Prolixity we make Choice of the moſt eaſy and ſimple Caſes. Which is what I would have all along obſerved. (d) 2. Lect. 7.

7. Let D be a given Point, and two Curves XEM, YFN ſo related, that drawing any right Line DEF from D; DE may have always the ſame Ratio to DF; and let the right Line FS touch one of the Curves YFN; to which let ER be parallel: Then will ER touch the Curve XEM. Fig. 81.

For draw the right Line DK any how from D (cutting the Lines as you ſee). Then it is DK : DI : : DF : DE : : DN : DM.

: D M. Therefore ſince it is D K ⊏ D N; it ſhall be D I ⊏ D M; and ſo the whole right Line R E will fall without the Curve X E M.

The right Lines N K, M I have always the ſame Ratio. Wherefore the Thing is otherwiſe evident.

8. Let three Curves X E M, Y F N,
Fig. 82. Z G O be ſuch, that any how drawing, from a given Point D, the right Line D E F G, the Intercepts E G, E F may have always the ſame Ratio, ſuppoſe that of R to S; and let E T, G T touch two of the Curves (ſuppoſe X E M, Z G O) in the Points E, G; it is required to draw a right Line to touch the Curve Y F N in the Point F.

Conceive the Curve T F V ſuch, that drawing any how from D the right Line D M L K (which cuts the right Lines T E, T G in the Points I, L, and the Curve itſelf in K) I L may have the ſame Ratio
(a) 2. Lect. 8. to I K as R to S. Then *(a)* is I K ⊏ I N; and ſo the Curve T F M touches the Curve
(b) 4. Lect. 6 Y F N. But *(b)* the Curve T F K is
(c) 2. of this. an *Hyperbola*, let F S touch this: Then will *(c)* F S touch the Curve Y F N too.

Becauſe we have here firſt happened to make mention of the Tangents of *Hyperbola*'s, we ſhall determine at the ſame Time not only that of the common *Hyperbola*, but thoſe too of all other *reci-*

procal Lines generated after the ſame way with that.

9. Let VD be a right Line, and the two Curves XEM, YFN ſo related, that drawing at pleaſure a right Line EDF parallel to a right Line given in Poſition, the Rectangle under DE, DF may be always equal to ſome given Space; and let the right Line ET touch the Curve XEM in E, meeting the right Line VD in the Point T; aſſume $DS = DT$; joyn FS: And this will touch the Curve YFN in F. Fig. 83.

For any how draw IN parallel to EF (cutting the Lines as *per* Scheme.) Then is $TP : PM \sqsubset (TP : PI ::) TD : DE$. Alſo $SP : PK :: SD : DF$. Therefore $TP \times SP : PM \times PK \sqsubset TD \times SD : DE \times DF :: TD \times SD : PM \times PN$. But $TD \times SD \sqsubset TP \times SP$; and thence $TD \times SD : PM \times PK \sqsubset TD \times SD : PM : PN$. Wherefore $PM \times PK \sqsupset PM \times PN$ or $PK \sqsupset PN$. And ſo the right Line FS will fall quite without the Curve YFN.

Obſerve, If XEM be a right Line *(viz.* co-inciding with TEI) YFN will be the common *Hyperbola*; T being its Centre, TS one of the *Aſymptotes*, and the other TZ will be parallel to EF.

10. Moreover let D be a given Point, and two Curves XEM, YFN ſo related, that drawing any right Line EF from D; the Rectangle under DE, DF may be
Fig. 84. perpetually equal to a given Square, ſuppoſe that of the Line Z. And let the right Line ER touch one of the Curves XEM; the Tangent to the other (at F) may be thus determined: Draw DP perpendicular to ER; and make DP : Z :: Z : DB; biſſect DB in C, joyn CF, and draw FS normal to CF; this will touch the Curve YFN.

For deſcribe a Circle DOB from C
thro' the Point F, and thro' B draw any
how the right Line IN (interſecting the
Lines as you ſee.) Then is DO × DI
(a) 27. *(a)* = DP × DB *(b)* = Z *q* *(c)* = DM
Lect 6. × DN or DO : DM :: DN : DI.
(b) Conſtr. *(c) Supp.* Therefore ſince DM *(c)* $\sqsupset$ DI; it ſhall
be DO $\sqsupset$ DN; and ſo the Circle DOB
will touch the Curve YFN. Wherefore
the right Line FS will touch the ſame
Curve YFN.

Fig. 85. 11. Let the Curves XEM, YFN be ſuch, that drawing any right Line FE parallel to a right Line given in Poſition, this may be always equal to the ſame Line; and let SF touch the Curve YFN; the right Line RE parallel to this will touch the other Curve XEM.

For any how drawing M K parallel to F E it is $NI \sqsupset (KI = FE =) NM$. Wherefore the Point I, will fall without the Curve X E M, *&c.*

The Line X E M is really no other than the Curve Y F N put in another Situation. This Trifle is only propoſed for Method's ſake.

Fig 86.

12 Let X E M be any Curve, which the right Line E R touches in E; alſo let Y F N be another Curve ſo related to the other, that drawing the right Line D E F any how from a given Point D, the intercepted Part E F may be always equal to ſome determinate Line Z; it is required to draw a right Line to touch the Curve Y F N in the Point F. Take $DH = Z$, and thro' H draw A H perpendicular to D H, meeting E R in the Point B, and thro' F draw F G perpendicular to A B, and aſſume $GL = GB$; then if you draw L F S it will touch the Curve Y F N.

For conceive an *Hyperbola* O F O to be deſcribed thro' F to the *Aſymtotes* E R, A B, which any right Line D O drawn from D cuts. Then is *(a)* $QO = DP$; and ſo $MO \sqsubset DP$ *(b)* $\sqsubset DH$ *(c)* $= MN$. Therefore the *Hyperbola* O F O touches the Curve Y F N.

But the right Line L S *(d)* touches the *Hyperbola* O F O: Wherefore this will touch the Curve Y F N too.

(a) *Inverſe* 9. Lect. 7.

(b) *Supp.*

(c) *Elem.*

(d) 9. of this.

Note, If XEM be ſuppoſed a ſtrait Line (or coincides with ER); YFN will be the firſt Common, *Conchoid*, *viz.* that of *Nicomedes*. And therefore you have the Determination of its Tangent, from a certain general Method.

Fig. 87. Let LA be a right Line and BEI any Curve; as alſo let there be ſome other Curve DFG ſuch, that drawing at Pleaſure the right Line PFE parallel to a right Line given in Poſition, let the Square of the right Line PE be equal to the Square of PF together with the Square of the given Line Z; alſo let the right Line ET touch the Curve BEI; and then make $PEq : PFq :: PT : PS$. This done, if SF be join'd; it will touch the Curve DFG.

For conceive the Curve VFH to be ſuch, that drawing QK parallel to PE (cutting the given Lines as you ſee) QKq may be perpetually equal to $QHq + Zq$: Whence, becauſe it is (*a*) $QK ⊏ QI$; it ſhall be $QKq - Zq ⊏ QIq - Zq$; that is, $QHq ⊏ QGq$: Therefore the Curve VFH will touch the Curve DFGF. But the Curve VFH (*b*) is an *Hyperbola*, which the right Line SF touches (*c*): Wherefore this will alſo touch the Curve DFG.

(*a*) Supp.
(*b*) 22. Lect. 6.
Lect. 1.
(*c*) Cor. 22

14. If the ſame Things remain; but now $PEq + Zq = PFq$, and you make $PEq : PFq :: PT : PS$. Then FS (joyn'd) will again touch the Curve DFG. Fig. 88.

The Demonſtration is the ſame; but the 23^{d} of the firſt Lecture muſt be taken.

15. Let two Curves AFB, CGD, having the common Axis AD, be ſo related to one another, that drawing any right Line FEG perpendicular to AD (cutting the given Lines, as *per* Figure) the Sum of the Squares of EF, EG may be equal to the Square of a determinate Line Z; and let the right Line FR touch one of the Curves, as AFB. Then if you make $EFq : EGq :: ER : ET$, and joyn GT. This will touch the Curve CGD too. Fig. 89.

For conceive the Curve OGO to be ſuch, that drawing the right Line KQO (which Interſects the right Lines ER, FR in the Points Q, K, and the Curve OGO in O) $QKq + QOq$ may be $= Zq$, and ſo it will be $QKq + QOq = QIq + QLq$; and becauſe it is QKq *(a)* $\sqsubset QIq$, it ſhall be $QOq \sqsupset QLq$. *(a)* Supp. Wherefore the Curve OGO will touch the Curve CGD inwardly. But *(b)* from what has been already demonſtrated, the Curve OGO is an *Ellipſis*, which the right *(b)* 24. Lect. 6.

Line

Line GT touches. Therefore GT will touch the Curve CGD too.

Fig. 90. 16. Let there be any Curve AFB (whoſe Axis is AD, and Ordinate to it DB) let alſo another Curve VGC be ſo related to it, that drawing the right Line ZF any how from a given Point Z in the Axis AD, and thro' F the right Line EFG parallel to DBC; we may have EG equal to ZF; alſo let PQ be perpendicular to the Curve AFB; and aſſume QR equal to ZF: Then if the Line GR be drawn, it will be perpendicular to the Curve VGC.

For draw FT perpendicular to FQ, or touching the Curve AFB; and conceive the Curve OGO to be ſuch, that drawing any right Line HKO parallel to EGF (cutting the right Lines TE, TF, and Curve OGO in the Points H, K, O) and joining ZK; we may have HO = ZK;
(a) Hyp. alſo draw ZI. Becauſe HK *(a)* $\sqsubset$ HI, it ſhall be ZK $\sqsubset$ ZI, or HO $\sqsubset$ HL: Wherefore the Curve OGO touches the
(b) 25. Lect. 6. Curve VGC. *(b)* But OGO is an *Hyperbola*, to which the right Line GR is perpendicular: Therefore the ſame GR will be alſo perpendicular to the Curve VGC.

Fig. 91. 17. Let DQ be a right Line, and the two Curves DRS, DYX ſo related, that any how drawing a right Line REY parallel to the right Line D B given in Poſition (which cuts the ſaid Lines as you ſee) and joyning D Y, it may be always R Y : D Y :: D Y : E Y; and let the right Line R F touch the Curve D R S in R. It is required to draw a right Line to touch the Curve D Y X in Y.

Conceive the Line D Y O ſuch, that any how drawing G O parallel to D B (which Interſects the Lines F R, F P, D Y O in the Points G, P, O) and joyning D O, it may be always G O : D O :: D O : P O: the Curve D Y O will touch the Curve D Y X in the Point Y. For let the right Line G O cut the Curves D R S, D Y X in the Points S, X, and joyn D G, D S, D X. Then it is evident (from the Nature of the Curve) the Angles X D P, D S P, as alſo the Angles O D P, D G P are equal; and ſo becauſe D S P is greater than the Angle D G P; the Angle X D P will be greater than the Angle O D P, and ſo P X greater than P O: Therefore the Curve D Y O will touch the Curve DYX in the Point Y. But the Curve D Y O (a) is an *Hyperbola*; let Y S touch it: Therefore this will alſo touch the Curve DYX.

(a) 12. Lect. 6.

Note, If the Curve D R S be a Circle, and Q D B a right Angle, the Curve DYX will be the common *Ciſſoid*; and ſo the Tangent to this, as well as innumerable Curves generated in like manner, is here determined.

18. Let

Fig. 92. 18. Let D B, B K be right Lines given in Poſition, and the Curve D Y X ſuch, that drawing any right Line D Y H from the Point D (which cuts B K in H, and the Curve D Y X in Y) the Subtenſe D Y may be always equal to the right Line B H. It is required to draw a right Line to touch the Curve D Y X in Y.

From the Centre D thro' B draw the Circle B R S, which the right Line Y E R drawn parallel to B K meets, and joyn D R. Then (becauſe the Ang. D Y E = Ang. D H B, and D Y = B H, and D R = B D) the Triangle R D Y is ſimilar and equal to the Triangle D B H. Wherefore R Y : Y D :: (D H : H B) :: Y D : Y E. Therefore (by our laſt) a right Line may be drawn to touch the Curve D Y X in the Point Y.

19. In like manner let D B, B K be Fig. 93. right Lines given in Poſition, and the Curve B X X ſuch, that drawing any right Line D X from D (which cuts the right Line B K in H, and the Curve B X X in X) H K may be always equal to B H. It is required to draw a right Line to touch the Curve B M X in the Point X.

Conceive the Curve D Y Y ſuch, that it may be always D Y = B H (*viz.* ſuch, as touched upon in our laſt) and let the right Line Y T touch this in the Point Y, meeting B K in the Point R. Then let an *Hyperbola*

bola NXN be deſcribed thro' X to the *Aſymtotes* RB, RT, to which any how draw the right Line DN (cutting the ſaid Lines as you ſee) Now OM (*a*) = DI) ⊐ (*a*) DL (*b*) =) ON. Therefore the *Hyperbola* NXN touches the Curve BXX in the Point X. And ſo draw the right Line XS touching the *Hyperbola* NXN. This will alſo touch the Curve BXX.

(*a*) *Conſtr.*
(*b*) *Inverſe* 9 Lect. 6.

We ſeem to have trifled ſufficiently for this Time, and ſhall now reſt a while.

LECTURE IX.

WE are forthwith moving directly forwards in the Way already entered upon.

Fig. 94. 1. Let the right Lines AB, VD be parallel to each other, and the right Line DB given in Poſition cut them; and let the Lines EBE, FBF paſſing thro' B be ſo related, that drawing any Line PG parallel to DB; PF may be always an *arithmetical Mean* of the ſame given Order between PG, PE, and let the right Line BS touch the Curve EBE; it is required to draw a right Line to touch the Curve FBF in the Point B.

Let the Numbers N, M be the Exponents of the Proportionals PF, PE (ſuch *(a)* as have been explained above;) make N : M :: DS : DT, and joyn TB: This ſhall

(a) 12. Lect. 7.

ſhall touch the Line F B F. For draw the right Line P G any how, cutting the ſaid Lines, as *per* Scheme. Then it is F G : E G : : *(b)* N : M : : *(c)* D S : D T (b) 11. Lect. 7.
: : *(d)* L G, K G; and ſo ſince *(e)* K G $\sqsupset$ (c) *Conſtr.*
E G, it ſhall be L G $\sqsupset$ F G: Conſequently (d) 3. Lect. 7.
the right Line T B falls quite without (e) *Supp.*
the Curve F B F.

2. Things remaining the ſame, let P F be now a geometrical mean Proportional (of the ſame Order it was before, when an arithmetical Mean) between P G, P E; The ſame Line B T will touch the Curve F B F.

For the Lines conſtructed after this manner (a) 17. Lect. 7.
from arithmetical Means and geometrical mean Proportionals *(a)* do touch one another in the Point B: Therefore becauſe the right Line B T touches the one, this will touch the other alſo.

Example. Let PF be the third of ſix Means between P G, P E: then will M $= 7$, and N $= 3$; and ſo D S : D T : : 3 : 7.

3. The Suppoſition in the laſt Number Fig. 95.
remaining, if any Point F be taken in the Curve F B F; a right Line may alſo be drawn after the ſame Manner to touch the Curve in this Point.

For thro' F draw the right Line P G parallel to D B, cutting the Curve E X E in E, let E X touch the Curve E B E in

E; make N : M :: P X : P Y; joyn the right Line F Y: this will touch the Curve F B F.

For draw the right Line C E thro' E, parallel to A B (or V D); and conceive the Curve H E H paſſing thro' E to be ſuch, that drawing any right Line Q L parallel to D E (cutting the Curves E B E, H E H in L and H, and the right Lines C E, V P in I and Q); Q H may be always a Mean of the ſame Order between Q I, Q L, as P F is between P G, P E. Then it is evident (from the laſt Number) that the right Line E Y (being joyn'd) touches the Curve H E H. But the Curve E B F is *(a)* analogous to the Curve H E H. Therefore *(b)* the right Line F Y will touch the Curve F B F too.

(a) 7. Lect. 7.

(b) 5. Lect. 8.

4. You may obſerve when E B E is a right Line, the Line F B F will be ſome one of the *Paraboloids*. And ſo what has been here and there obſerved concerning them, (deduced from Calculation and made manifeſt by a Sort of Induction, but not to my Knowledge, any where Demonſtrated geometrically) does flow in abundance from a moſt plentiful Spring diffuſing it ſelf to innumerable Curves of other kinds.

Fig. 96.

5. Hence it evidently follows, if T D be a right Line, and two Curves E E E, F F F be ſo related, that drawing the

right Lines PEF parallel to BD given in Poſition, let the Ordinates PE be always as the Squares of the Ordinates PF, and let the right Lines ES, FT (drawn from the Bounds of the ſame common Ordinate) touch theſe Curves: Then will TP = 2 SP; if the Ordinates PE be as the Cubes of PF; it ſhall be TP = 4 SP; and ſo on *ad Infinitum* according to this Law.

6. Let (again) there be a Circle ABC, whoſe Centre is D, and Radius DB, and let the Lines EBE, FBF paſſing thro' B be ſo related, that drawing any right Line DG thro' D; DF may be always an arithmetical Mean of the ſame Order between DG, DE; alſo let the right Line BO touch the Curve EBE in the Point B. It is required to draw a right Line to touch the Curve FBF in the ſame Point B. Fig. 97.

This is in ſome ſort generally effected, (N. 8. of the Eighth Lecture) But we ſhall perform the thing here more plainly and largely. Let DQ be perpendicular to DB, which BO cuts in S, and make N : M :: DS : DT; joyn the right Line TB. Then ſhall this touch the Curve FBF.

For let the right Line PB touch the Circle ABG, and ſo divide the Lines DS in X and BS in Y, that it may be DS : DX :: Fig. 97.

DX :: M:N :: BS:BY; and draw XZ, YV thro' the Points X, Y parallel to BS, DS, meeting one another in the Point C; conceive an *Hyperbola* LBL to be deſcribed thro' B to the *Aſymtotes* YCZ; alſo draw a right Line DP any how from D, interſecting the ſaid Lines, as *per* Scheme. Then is PK : PL :: *(a)* M:N :: *(b)* GE: GF *(c)* ⊏ PE:PF ⊏ PK:PF: Wherefore PL ⊐ PF; and ſo the *Hpperbola* LBL touches the Curve FBF. Now continue out TB meeting XZ in R; then it is RZ : ZB :: BS : ST. Conſequently RZ × ST = BS × ZB = BS × SX. But becauſe DS:SX :: *(d)* BS:SY. We have DS × SY = BS × SX; and ſo RZ × ST = DS × SY = DS × CX. Or RZ : CX :: DX : ST, and (by Compoſition) RZ : RZ + CX :: DS : DT :: *(d)* N:M :: CZ : CZ + CX. And ſo (by Diviſion) RZ : CX :: CZ : CX. Therefore RZ = CZ. Wherefore RB touches the *Hyperbola* LBL. Whence RBT will touch the Curve FBF, contiguous to LBL.

(a) Inverſe of 4. Lect. 6.
(b) 11. Lect. 7.
(c) 1. Lect. 7.
(d) Conſtr.

7. Hence if (things remaining the ſame) the right Line DF only be perpetually made a geometrical Mean between DG, DE, of the ſame Order as before; the ſame BT will touch the Curve FBF alſo.

For

Plate IV.
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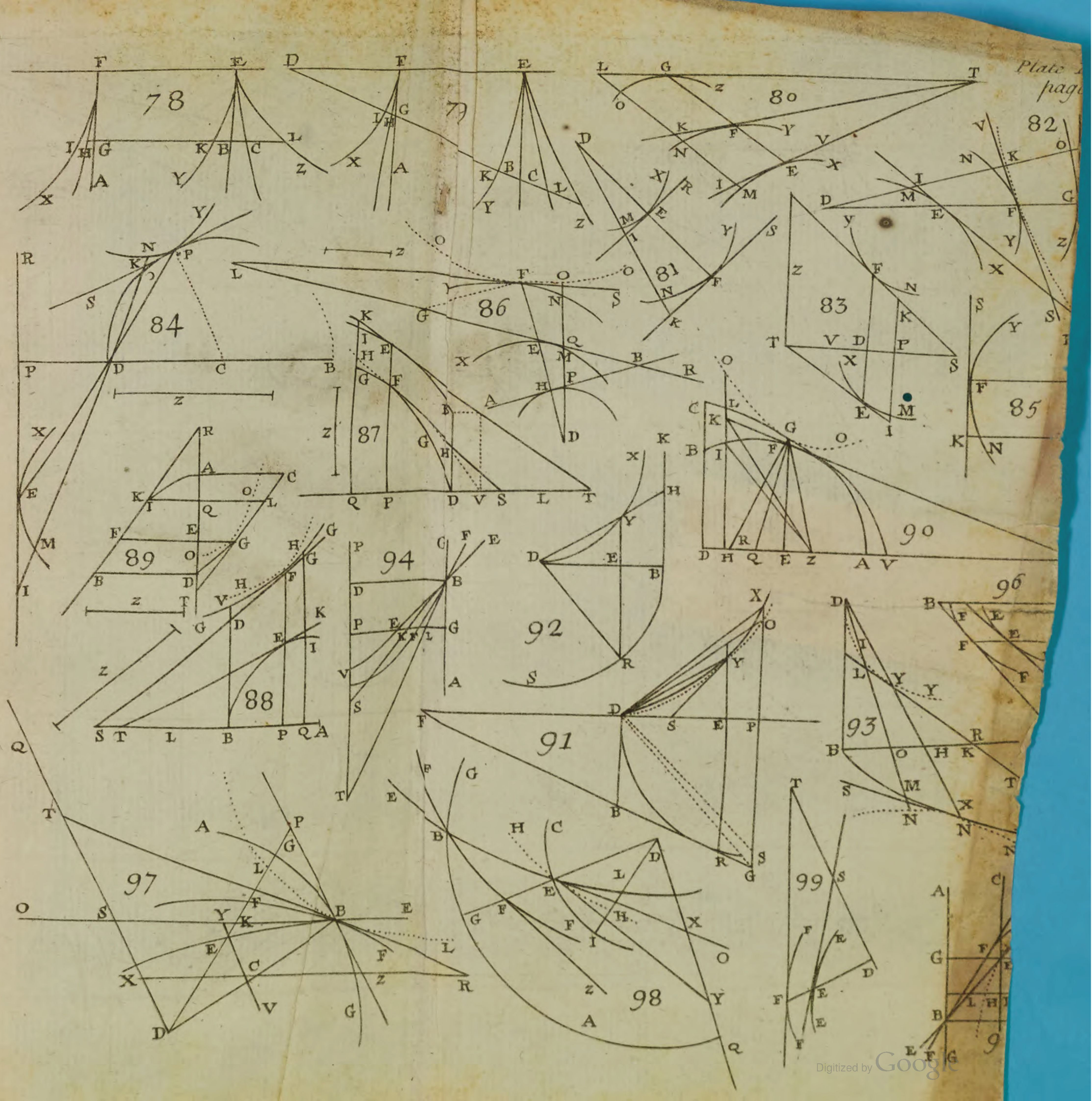

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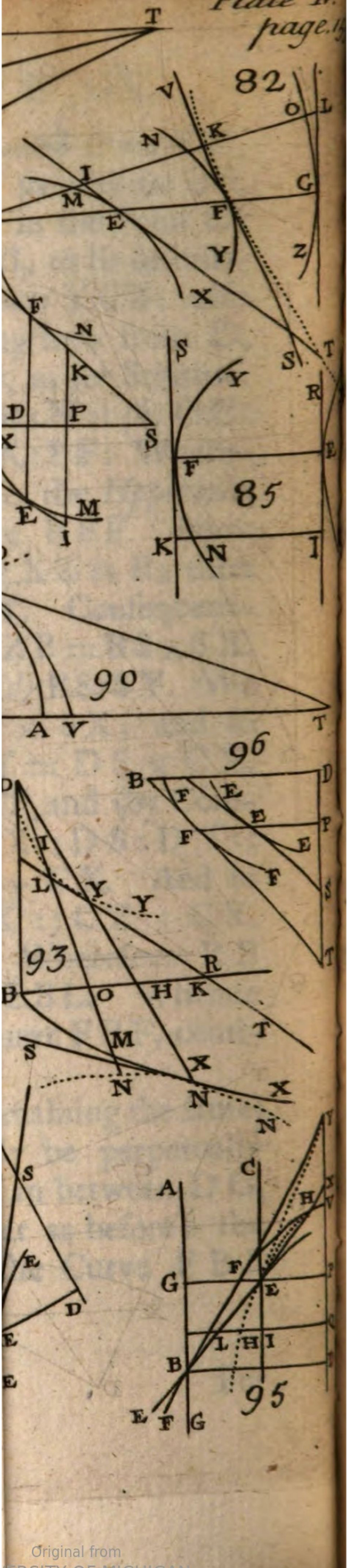

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For Lines formed from arithmetical Means and geometrical mean Proportionals, of the ſame Order, do touch one another; and ſo one right Line will touch them both.

8. Moreover (things remaining as in our laſt) a right Line may in like manner be drawn from any given Point F in the Curve FBF. Fig. 98.

For joyn the right Line DF interſecting the Curve EBE in the Point E; as alſo draw DQ perpendicular to DG cutting EO in X; make DX : DY :: N : M; joyn EY, and draw FZ parallel to EY; this will touch the Curve FBF.

For with the Centre D draw a Circle CEI thro' E; conceive the Line HEH to be ſuch, that drawing from D any right Line DI (cutting the Circle CE in I, the Curve HEH in H, and EBE in L) DH may be always a mean Proportional of the ſame Order between DI, DL, as DF is between DG, DE; then it is plain (from our laſt) that the right Line EY will touch the Curve HEH. But the Curve FBF *(a)* is analogous to the Curve HEH: Wherefore *(b)* the right Line FZ will alſo touch the Curve FBF. (a) 9. Lect. 7. (b) 7. Lect. 8.

Hence we may very ſpeedily determine not only the Tangents of innumerable Spirals, but thoſe of infinite Numbers of others of different Kinds.

9. Hence

Fig. 99. 9. Hence it is manifeſt, if two right Lines EEE, FFF be ſo related to each other, that any how drawing right Lines DEF from the Point D; the right Lines DE may be to each other as the Squares of DF, at the Ends of which let the right Lines ES, FT touch the Curves, meeting the Perpendiculars to DEF in the Points S, T: Then will always DT = 2 DS. If DE be as the Cubes of DF; it will always be DT = 3 DS. And ſo on after the ſame Law.

Fig. 100. 10. Let the right Lines VD, TB meet one another in the Point T, and the right Line DB given in Poſition cut them; alſo let the Lines EBE, FBF, paſſing thro' B be of ſuch a Nature, that drawing any Line PG parallel to DB; PF may be always an arithmetical Mean of the ſame Order between PG, PF; and let BR touch the Curve EBE; it is required to draw a right Line to touch the Curve FBF in the Point B.

Take NM (the Exponents of the Orders in which are PF, PE) and make

$$N \times TD \left.\begin{array}{l} + M \\ - N \end{array}\right\} \times RD : M \times TD$$

:: RD : SD; joyn BS, and this will touch the Curve FBF.

For any how draw PG, cutting the ſaid Lines as *per* Scheme. Then is EG:

$FG :: (a) M : N$. And therefore $FG \times TD : EG \times TD :: N \times TD : M \times TD$. Alſo $EF \times RD : EG \times TD :: M - N \times RD : M \times TD$. Wherefore (joyning the Antecedents) it ſhall be $FG \times TD + EF \times RD : EG \times TD :: N \times TD + M - N \times RD : M \times TD$, (that is) $:: (b)\ RD : SD$. But (c) $LG \times TD + KL \times RD : KG \times TD :: RD : SD$. Therefore $FG \times TD + EF \times RD : EG \times TD :: EG \times TD + KL \times RD : KG \times TD$. And ſo, ſince $EG\ (d) \sqsubset KG$; it ſhall be $FG \times TD + EF \times RD \sqsubset LG \times TD + KL \times RD$. Or $FG : EF + TD : RD \sqsubset LG : KL + TD : RD$; or (taking away the common Ratio) $FG : EF \sqsubset LG : KL$. Or (*Componendo*) $EG : EF \sqsubset KG : KL\ (e) \sqsubset EG : EL$. Whence it is $EF \sqsupset EL$. And ſo the Point L is ſituate without the Curve FBF. Conſequently the Thing propoſed is manifeſt.

(a) 11. Lect. 7.
(b) Conſtr.
(c) 4. Lect. 7.
(d) Hyp.
(e) 1. Lect. 7.

11. Things remaining as before, if PF be ſuppoſed a geometrical Mean of the ſame Order, it follows (exactly as in the Method above) that the ſaid Line BS will touch the Curve FBF.

Example. If PF be the third of ſix Means, or $M = 7$, and $N = 3$; it ſhall be $3\,TD + 4\,RD : 7\,MD :: RD : SD$, or

$$SD = \frac{7\,MD \times RD}{3\,TD + 4\,RD}.$$

12. It

Fig. 101. 12. It is alſo evident, that a right Line may be drawn after the ſame Manner from a given Point (as F) in the Curve F B F to touch it. For draw the right Line P G thro' F parallel to D B, cutting the Curve EBE in E; thro' E draw ER touching the Curve E B E, and make $N \times TP \left.\begin{matrix} +M \\ -N \end{matrix}\right\} \times RP : M \times TP :: RP : SP$, and joyn S F: this will touch the Curve F B F. Being demonſtrated after the very ſame Manner as the 3*d* Number of this Lecture; only here (a Parallel is not drawn to V D thro' E but E T is joyn'd) and the eighth Number of the ſeventh Lecture is of uſe.

13. Obſerve, if E B E be a right Line (*viz.* co-inciding with the right Line BR) the right Line F B F is ſome one of the *Hyperboliform* Curves, the Method of drawing Tangents to which, as well as to an infinite Number of Curves of other Kinds, we have laid down in one *Theorem*.

Fig. 102. 14. If the Points T, R do not both fall on the ſame ſide of D or P; the Tangent (B S) of the Curve F B F will be determined by making $N \times RD \left.\begin{matrix} +M \\ -N \end{matrix}\right\} \times TD : M \times TD :: RD : SD$.

This is evident by the very ſame Way of Reaſoning, and from the fifth Number of the ſeventh Lect. inſtead of the fourth.

15. Hence may be determined at once, not only the Tangents to all kinds of *Elliptick Figures*, (ſuppoſing EBE to be a right Line, coinciding with the Line BR) but likewiſe thoſe to innumerable Curves of other kinds.

Example. Let PF be the fourth of four Means, or $M=5$, and $N=4$. Then will be $SD = \frac{5\,TD \times RD}{4\,RD - TD}$.

Obſerve. If it ſhould happen that $ND \times RD = \left.\begin{matrix} + M \\ - N \end{matrix}\right\} \times TD$; DS would be infinite, or BS parallel to VD. Other Obſervations may be made, but I ſhall leave them.

Among innumerable other Curves, the Fig. 103.
Ciſſoid and all kind of Ciſſoidal Curves have their Tangents determinable by this Method. As ſuppoſe DSB be half a right Angle, and the two Curves SGB, SEE, ſo related to one another, that drawing at pleaſure the right Line GE parallel to BD (cutting the given Lines, as you ſee;) PG, PF, PE be continual Proportionals, and the right Line GT touches the Curve SGB in G; you will find the Line touching SEE in the Point E, in making $2TP - SP : TP :: SP : RP$. For when the Points R, E are joyn'd, it will touch the Curve SEE; as is eaſy to infer from what has been already ſaid. Now

Now if the Curve SGB be a Circle, and the Angle SPG of Application be a right Angle, the Curve SEE is the *common* Ciſſoid, *viz.* that of *Diocles*; if not, it is a Ciſſoid of another Kind. We have juſt touched upon this by the Way, and ſhall now detain you no longer.

LECTURE X.

WE are ſtill continuing on the Buſineſs of Tangents.

1. Let there be two Curves AEG, AFI ſo related to one another, that drawing any right Line EF, parallel to AB given in Poſition, cutting the Curve AEG in E, and the Curve AFI in F; EF be always equal to the Arch AE of the Curve AEG, reckoning from A; and the right Line ET touches the Curve AEG in the Point E, and FT be made equal to the Arch AE: If TF be drawn, it will touch the Curve AFI. Fig 104.

For draw the right Line GK any how parallel to AB, cutting the given Lines, as you ſee. Then is $GK = GH + HK = GH + HT$ *(a)* $=$ Arch $AG = GI$.

(a) 22. Lect. 7.

Whence the Point K is ſituate without the Curve A F I; and ſo the right Line T K touches it.

2. If the right Line E F has any given Ratio to the Arch A E, the right Line F T will nevertheleſs touch the Curve A F I; as evidently follows from hence and the ſixth Number of the eighth Lecture.

The preſent *Theorem* we have ſome time ago demonſtrated otherwiſe. But this Demonſtration here, ſeems to be ſomewhat more ſimple and evident; and more ſuitable to our Method in hand.

Fig. 105. 3. Let there be any Curve A G E, and the given Point D; alſo let A I F be another Curve ſuch, that drawing any right Line D E F from D, the intercepted Part E F be always equal to the Arch A E, and let the right Line E T touch the Curve A G E; it is required to draw a Tangent to the Curve A I F in F.

Make T E = the Arch A E, and let the Curve T K F be ſuch, that drawing the right Line D K any how from D (cutting the Curve T K F in K, and the right Line T E in H) it may be always H K = H T. Then let the right Line F S touch the Curve (a) T K F in the Point F. This will alſo touch the Curve A I F.

(a) 17. Lect. 8.

For GK is = GH + HK = GH + HT, and (b) GA = GI. Conſequently, the Point K falls without the Curve AIF; and ſo the right Line FS touches the Curve AIF. (b) 22. Lect. 7.

4. If the right Line EF be ſuppoſed to have any given Ratio to the Arch AE, its Tangent may be eaſily determined by this and the eighth Number of the eighth Lecture.

Fig. 106.

5. Let there be a right Line AP, and two Curves AEG, AFI ſo related to each other, that any how drawing the right Line DEF, cutting the right Line AP, and the Curves AEG, AFI in the Points D, E, F; it may be always, the right Line DT equal to the Arch AE; alſo let the right Line ET touch the Curve AEG in E; take ET equal to the Arch EA, let TR be parallel to BA, and joyn the Points R, F: The Line RF will touch the Curve AFI.

For conceive the Curve LFL to be ſuch, that drawing any right Line PL parallel to AB, (cutting the Curve AEG in G, the right Line TE in H and the Curve LFL in L) the right Line PL may be perpetually equal to TH, HG together; * Supp. then is (b) $\text{PL} \sqsubset \text{AEG}^{*} = \text{PI}$. Wherefore the Curve LFL touches the Curve AFI. (c) 26. Lect. 6. Alſo the right Line IK (c) is equal to the right Line TH; and ſo the right Line

(d) 3. Lect. 8. R F K (d) touches the Curve L F L; and
(e) 2. (e) conſequently the Curve A F I.

Lect. 8. 6. Alſo if the right Lines D E have a conſtant Ratio to the Arch A E; the right Line R F will nevertheleſs touch the Curve A F I; as is eaſily manifeſt from hence, and and the ſixth Number of the eighth Lecture.

7. Let D be a given Point, and two
Fig. 107 Curves A G E, D I F be ſo related to each other, that drawing any right Line D F E from the Point D, the right Line D F may be equal to the Arch A E, and let the right Line E T touch the Curve A G E in the Point E; it is required to draw a right Line D I F to touch the Curve D I F in the Point F.

Aſſume E T equal to the Arch F S, and conceive the Curve D K K to be ſuch, that drawing the right Line D H any how from D (cutting the Curve D K K in K, and the right Line T E in H,) D K may
be perpetually equal to T H. Then let (a)
(a) 16. the right Line F S touch the Curve D K K
Lect. 8. in F; this will alſo touch the Curve D I F.

For let the Curve L F L be ſuch, that drawing any right Line D H from D (which cuts
(b) 22. the right Line T E in H, and the Curve
Lect. 7. L F L in L) it may be always D L $=$
(c) Hyp. T H + H G; then is D L (b) $\sqsubset$ Arch A G
(d) 4. Lect. 8. (c) $=$ D I; and ſo (d) the Curves D I F,
(e) 2. Lect. 8. L F L (b) will touch one another. Alſo the

the Curves KFK, LFK do touch one another. Wherefore the Curves DIF, KFK will alſo touch one another. Therefore the right Line FS will touch the Curve DIF.

8. If the right Lines DF have any given Ratio to the Arch AE; the Tangent to the Curve DIF may be determined from hence, and the ſecond Number of the laſt Lecture; it being parallel to the Tangent FS.

9. Hence may be determined not only the *Tangents* to *circular Spirals*, but alſo to infinite Numbers of other Curves generated after the ſame Manner. Fig. 108.

10 Let AEH be any Curve, AD any right Line (in which is the determinate Point D) and DH a right Line given in Poſition: Alſo let the Curve AGB be ſuch, that taking any Point G in it, and drawing thro' G and D the right Line DGE (cutting the Curve in E,) as alſo drawing GF parallel to DH; AE may have to AF the given Ratio of X to Y; and let the right Line ET touch the Curve AEH; it is requir'd to draw a right Line to touch the Curve AGB in G.

Make the right Line EV equal to the Arch EA, and conceive the Curve OGO to be ſuch, that drawing any right Line DOL (cutting the Curve OGO in the Point O, and the right Line ET in L)

and

and OQ parallel to GF; let VL : AQ :: X : Y; then (from what has been above demonſtrated) the Curve OGO will be an *Hyperbola*. Let the right Line GS touch this: Then will it touch the Curve AGB too.

For conceive another Curve NGN ſuch, that when a right Line DL drawn at pleaſure cuts it in N, the Curve AEH in K, the right Line TE in L, and when NR is drawn parallel to GF, it may be VL + LK : AR :: X : Y; then it is manifeſt that the Curve NGN does touch both the Curves AGB, OGO. [For let the right Line DL cut the Curve AEB in I, and draw IP parallel to GF: Then ſince VL + LK : AR :: X : Y :: AK : AP, and it is VL + LK $\sqsubset$ AK; it ſhall be AR $\sqsubset$ AP, or DR $\sqsupset$ DP; and ſo DN $\sqsupset$ DI. Whence the Point N will always fall within the Curve AGB; and ſo the Curve NGN will touch the Curve AGB.] Therefore the Curves AGB, OGO do touch one another. Conſequently becauſe the right Line GS touches the Curve OGO; it will touch the Curve OGB too.

Let the Curve AEH be a Quadrant of a Circle and D the Centre; the Curve AGB will be the common *Quadratrix*. Therefore the Tangent of it may be determined after this manner, as may thoſe too of all Curves generated like it.

I might here inſert ſeveral other Thoughts on this Subject, but in my Opinion, theſe are ſufficient to explain the Method by which Tangents to Curves may be found without any troubleſome Calculations, and the Conſtructions demonſtrated at the ſame Time. I ſhall only ſubjoyn one or two general Theorems, not to be deſpiſed.

11. Let there be any Line Z G E, whoſe Axis is V D; and firſt, let the perpendicular Ordinates V Z, P G, D E to it, any how increaſe continually from the firſt V Z; alſo let the Line V I F be ſuch, that drawing any right Line E D F perpendicular to V D (which cuts the Curves in the Points E, F, and V D in D,) the *Rectangle* under D F and any given right Line R, may be reſpectively equal to the intercepted *Space* V D E Z; and make D E : D F :: R : D T, and joyn the right Line T F: this will touch the Curve V I F. Fig. 109.

For take any Point I in the Line V I E (firſt above the Point F, towards the beginning V) and thro' this draw the right Lines I G, K L parallel to V Z, V D (cutting the Lines, as *per* Scheme) then it is L F : L K :: (D F : D T ::) D E : R; and ſo L F × R = L K × D E. But (by Suppoſition) L F × R is equal to the Space P D E G. Fig. 110.

PDEG. Therefore $LK \times DE = PDEG \sqsupset DP \times DE$. Whence it is $LK \sqsupset DP$, or $LK \sqsupset LI$.

Again take any Point I, below the Point F, and make the reſt as before. Then by the like Reaſoning it will appear, that $LK \times DE = PDEG \sqsubset DP \times DE$; and ſo it will be $LK \sqsubset DP$, or LI. From whence it is manifeſt that the whole right Line TKFK falls within (or without) the Curve VIFI.

Things remaining as before, if the Ordinates VZ, PG, DE, *&c.* do continually decreaſe; the Concluſion and Reaſoning will be the ſame, with this difference only, that the Line VIF is Concave towards the Axis VD.

Coroll. We may obſerve that $DE \times DT =$ ſpace VDEZ.

Fig. 111. 12. From hence is this Theorem manifeſt. Let two Lines ZGE, VKF be ſo related, that any right Line EDF being made an Ordinate to the common Axis VD; we have always the *Square* of DE equal to twice the *Space* VDEZ. Take $DQ = DE$, and joyn FQ: this will be perpendicular to the Curve VKF.

For conceive the Line VIF, paſſing thro' F, to have the Property but juſt mentioned (*viz.* whoſe Ordinates to the Axis be to one another as the Space VDEZ, that is, in the preſent Suppoſition, as the

Squares of the Ordinates of the Curve V K F) and let the right Line F T touch the Line V I F; alſo let the right Line F S touch the Line V K F. Then is S D *(a)* = 2 T D. But D E × D T *(b)* = V D E Z. Therefore D E × S D = (2 V D E Z =) F D*q*. Conſequently Q F S is a right Angle.

(a) 5. Lect. 9.
(b) *Cor. aforeg.*

To theſe we will add the following *Theorem* of the ſame Kind.

13. Let there be any Curve A G E Z, and ſome Point D (from whence are drawn the Lines D A, D G, D E, *&c.* continually decreaſing from the firſt D A;) as alſo another Curve D K E (interſecting the firſt in E) of ſuch a Nature, that any how drawing from D a right Line D K G (cutting the Curve A E Z in G, and the Curve D K E in K) the Rectangle under D K and ſome ſtanding Line R, may be always equal to the Space A D G; and drawing D T perpendicular to D E, let D T = 2 R, and joyn T E: this will touch the Curve too.

Fig. 112.

For taking any Point K in the Curve D K E, draw the right Line D K G; and aſſuming D L = D K, draw L R parallel to D T (cutting D G in the Point Y.) Then thro' E draw E X perpendicular to D E (which will fall towards Z without the Curve, becauſe the Lines drawn from D towards Z do decreaſe; and ſo E X will fall

Fig. 113. fall within the Curve E G A towards A; ſo far at leaſt, as anſwers our purpoſe.) Now let the Point G, in the firſt place, be above E towards the beginning A. Then becauſe $TD : DE :: RL : LE$, and ſo $RL \times DE = TD \times LE$*(a)* $= 2R \times LE$*(a)* $= 2GDE \sqsubset 2DEX = EX \times DE$. Therefore $RL \sqsubset EX \sqsubset LY$. But the Point Y is without the Curve, becauſe $DY \sqsubset DL = DK$: Conſequently the Point R falls more without the Curve.

(a) Supp.

Let the Point G be below the Point E towards Z. Then it is again, as at firſt, $RL \times DE = GDE \sqsupset 2$ Triang. $EDX = EX \times DE$. Wherefore again $RL \sqsupset EX \sqsupset LY$. But the right Line L Y falls quite without the Curve E K (for if the Arch I K of the Curve K E be moved round, it will fall quite without.) Therefore the Point R does again fall without the Curve. Conſequently the right Line T E R touches the Curve D K E. And if any other Point, ſuppoſe K, be taken in the Curve D K E, D K G be drawn thro' it; and you make $DG : DK :: R : P$; aſſume $DT = 2P$; joyn T G; and draw K S parallel to G T: the right Line K S will touch the Curve D K E.

For conceive the Curve D O G, paſſing thro' G, to be ſuch that drawing any right Line D O N from D (which Interſects the Curve D O G in O, the Curve D N E in

M, and the Curve AGE in N) we may always have $DO \times P$ equal to the Space ADN. Then ſhall $DM \times R = DO \times P$; and ſo $DM : DO :: P : R$. Wherefore the Lines DKE, DOG are analogous. But (from what has been already demonſtrated) GT touches the Curve DOG: Therefore KS will touch the Curve DKE.

Obſerve that it is $DGq : DKq :: 2R : DS$.

For it is $DGq : DKq = DG : DK + DG : DK = R : P + DT : DS = R : P + 2P : DS = 2RP : P \times DS = 2R : DS$. Therefore $DGq : DKq :: 2R : DS$.

This is alſo true, and demonſtrated after the ſame Manner, when the right Lines DA, DG, DE, *&c.* drawn from D are equal (in which Caſe the *Curve* AGEZ ſhall be a *Circle*, and the Curve DKE the *Spiral* of *Archimedes)* or do conſtantly decreaſe from DA.

Hence may be eaſily deduced the following *Theorem.*

14. Let two Curves AGE, DKE be ſo related to each other, that drawing the right Lines DA, DG from a given Point D in the Curve DKE (the latter of which cuts DKE in the Point K) the *Square* of DK may be four times the *Space* ADG; drawing DH perpendicular to DG; ma- Fig. 114.

king D K : D G :: D G : D H; and joyning H K : this Line H K ſhall be perpendicular to the Curve D K E.

For conceive the Line D O K E, paſſing thro' K, to be of ſuch a Nature that Lines (ſuch as we have mentioned in our laſt) as D K, drawn from D to it may have the ſame Ratio as the Space A D G; and let the right Line K T touch the Curve D O K; the right Line K S the Line D K E; theſe meeting H D in the Points T, S. Then (by Prop. Preced.) $DGq : DKq :: \frac{DK}{2} : DT$. That is, (becauſe from what has been demonſtrated * $DS = 2\,DT$,) $DH : DK :: \left(\frac{DK}{2} : \frac{DS}{2} ::\right) DK : DS$. Therefore it is manifeſt the right Line H K is perpendicular to the Tangent K S.

* in 12. of this.

Thus I have in ſome ſort accompliſhed the chief Part of my propoſed Deſign. As a Supplement to which, I ſhall annex our Method of determining Tangents by Calculation. Tho' I ſcarcely perceive the Uſe of ſo doing, conſidering the ſeveral Methods of this Nature now become common and publiſhed. I do this at leaſt by the Advice of a Friend; and indeed ſo much the more willingly, as it ſeems to be compendious and general, with reſpect to what elſe I have handled. The Thing is thus.

Let AP, PM be right Lines given in Fig. 115. Poſition (whereof PM cuts the propoſed Curve in M,) and let MT touch the Curve in M, and cut the right Line AP in the Point T. Now to determine the length of the right Line PT, I ſuppoſe the Arch MN of the Curve to be indefinitely ſmall, and draw the right Lines NQ, NR parallel to MP, AP; I call MP, m; PT, t; MR, a; NR, e; and give Names to other Lines uſeful to our purpoſe, determin'd from the particular Nature of the Curve; and then compare MR, NR expreſſed by Calculation in an Equation, and by their means MP, PT themſelves; obſerving the following Rules at the ſame Time.

1. I reject all the Terms in the Calculation, affected with any Power of a or e, or with the product of them; for theſe Terms will be equal to nothing.

2. After the Equation is formed, I reject all the Terms wherein are Letters expreſſing conſtant or known Quantities; or which are not affected with a, or e; for theſe Terms brought over to one ſide of the Equation will be always equivalent to nothing.

3. I ſubſtitute a for m (M P), and t (P T) for e; by which means the Quantity of P T will be found.

When any indefinitely ſmall Particle of the Curve enters the Calculation, I ſubſtitute in its ſtead a Particle of the Curve properly taken; or any right Line equal to it, becauſe of the indefinitely Smallneſs of the Part of the Curve.

All this will appear more evident by the following Examples.

EXAMPLE I.

Fig. 116 Let A B H be a right Angle, and let the Curve A M O be ſuch, that drawing any right Line A K thro' A, cutting the right Line B H in K, and the Curve A M O in M, the Subtenſe A M may be equal to the Abſciſs B K; it is required to draw the Tangent (at M) of this Curve, or find the Value of the right Line P T.

Proceed according to the Directions above, and (drawing A N L) call A B, r, and A P, q. Then A G $= q - e$; alſo Q N $= m - a$. Therefore it is $qq + ee - 2qe + mm + aa - 2ma =$ (A Q q + Q N q) = A N q) = B L q; that is, (rejecting according to the Rule above) $qq - 2qe + mm - 2ma =$ B L q. Again it is A Q : Q N : : A B : B L; that is, $q - e : m - a :: r : BL = \frac{rm - ra}{q - e}$. Wherefore $\frac{rrmm + rraa - 2rrma}{qq + ee - 2qe} = BLq$. Or (caſting away what is ſuperfluous) $\frac{rrmm - 2rrmm}{qq - 2qe} = BLq = qq - 2qe + mm - 2ma$. Or $rrmm - 2rrma = q^4 - 2q^3e + qqmm - 2qqma - 2q^3e + 4qqee - 2qmme + 4qmae$, that is (rejecting as *per* Rule) $-2rrma = -4q^3e - 2qqma - 2qmme$, or $rrma - qqma = 2q^3e + qmme$. Or at length ſubſtituting m for a, and t for e, it is $rrmm - qqmm = 2q^3t - qmmt$, or $\frac{rrmm - qqmm}{2q^3 - qmm} = t = PT$.

EXAMPLE II.

Fig. 117.

Let EA be a right Line given in Poſition and Magnitude, and the Curve EMO of ſuch a Nature that drawing any how from it the right Line MP perpendicular to EA, the Sum of the Cubes of AP and MP may be equal to the Cube of the right Line AE.

Let $AE = r$, and $AP = f$. Then is $AQ = f + e$. And AQ Cub. $= f^3 + 3ffe + 3fee + e^3$, or (throwing away Superfluities, as *per* direction above) $= f^3 + 3ffe$. Alſo N Q Cub. = Cub. $m - a = m^3 - 3mma + 3maa - a^3$ (that is) $m^3 - 3mma$. Wherefore $f^3 + 3ffe + m^3 - 3mma$ is = (A Q Cub. + N Q Cub. = A E Cub. =) r^3. And caſting off given Quantities, it is $3ffe = 3mma = o$. Or $ffe = mma$, and putting m and t, for a and e, it will be $fft = m^3$, or $t = \frac{m^3}{ff}$. Therefore P T is a fourth Proportional in the continu'd Ratio of A P to P M.

In like manner, if it be A P qq + M P qq = A E qq; you will find P T to be $= \frac{m^4}{f^3}$, or P M a fourth Proportional in the Ratio of A P to P M; and ſo on. I do not know whether it be worth while to make this Obſervation of theſe *Cycloform* Lines.

EXAMPLE III.

Fig. 118. Let the right Line A Z be given in Poſition, and A X in Magnitude; alſo let the Curve A M O be ſuch, that any how drawing the right Line M P perpendicular to A Z, it may be A P Cub. + P M Cub. = A X × A P × P M.

Let AX $= b$, and AP $= f$, then is AQ $= ff - ee$, and AQ Cub. $= f^3 - 3ffe$, and QN Cub. $= m^3 - 3mma$, and AQ × QN $= fm - fa - me + ae = fm - fa - me$. Whence AX × AQ × QN $= bfm - bfa - bme$; hence the Equation $f^3 - 3ffe + m^3 - 3mma = bfm - bfa - bme$, and (rejecting and ſubſtituting) $bfa - 3mma = 3ffe - bme$, and $bfm - 3m^3 = 3fft - bmt$, or $\frac{bfm - 3m^3}{3ff - bm} = t$.

EXAMPLE IV.

Let CMV be a *Quadratrix*, (belonging to the Circle CEB whoſe Centre is A) and the Axis VA, and let the Ordinates AC, PM be perpendicular to the Axis VA. Fig. 119.

Continuing out the right Lines AME, ANF, and drawing the right Lines EK, LF perpendicular to AB, let the Arch CB $= p$, the Radius AC $= r$, the right Line AP $= f$, and AM $= k$. Now it is CA : Arch CB :: NR : Arch FE. That is, $r : p :: a : \frac{pa}{r} =$ Arch FE. And AM : MP :: AE : EK. That is, $k : m :: r : \frac{rm}{k} =$ EK. Alſo AE : EK ::

Arch FE : LK. That is, $r : \frac{rm}{k} :: \frac{pa}{r} : \frac{pma}{rk} =$ LK. But AM : AE :: AP : AK; that is, $k : r :: f : \frac{rf}{k} =$ AK. Thereſore $\frac{rf}{k} - \frac{pma}{rk} =$ AL. And $\frac{rrff}{kk} - \frac{2fmpa}{kk}$ (taking away what is ſuperfluous) $=$ ALq; and ſo LFq $= \frac{rrkk - rrff + 2fmpa}{kk} = \frac{rrmm + 2fmpa}{kk}$.

But it is AQq : QNq :: ALq : LFq; that is, $\overline{f-e}^{2} : \overline{m+a}^{2}$ :: ALq : LFq. That is, $ff - 2fe : mm - 2ma :: rrff - 2fmpa : rrmm + 2fmpa$. Whence (rejecting as *per* Rule) we get $ffpa + mmpa - rrfa = rrme$. Or $kkpa - rrfa = rrme$, or (ſubſtituting as directed) $kkpm - rrfm = rrmt$, or $\frac{kkp}{rr} - f = t$. Therefore the right Line AT $= \frac{kk}{rr}p$; that is, (becauſe from the known Nature of the

Curve, AV is $=\frac{rr}{p}$) it ſhall be AT $=\frac{AMq}{AV}$, or AV : AM :: AM : AT.

EXAMPLE V.

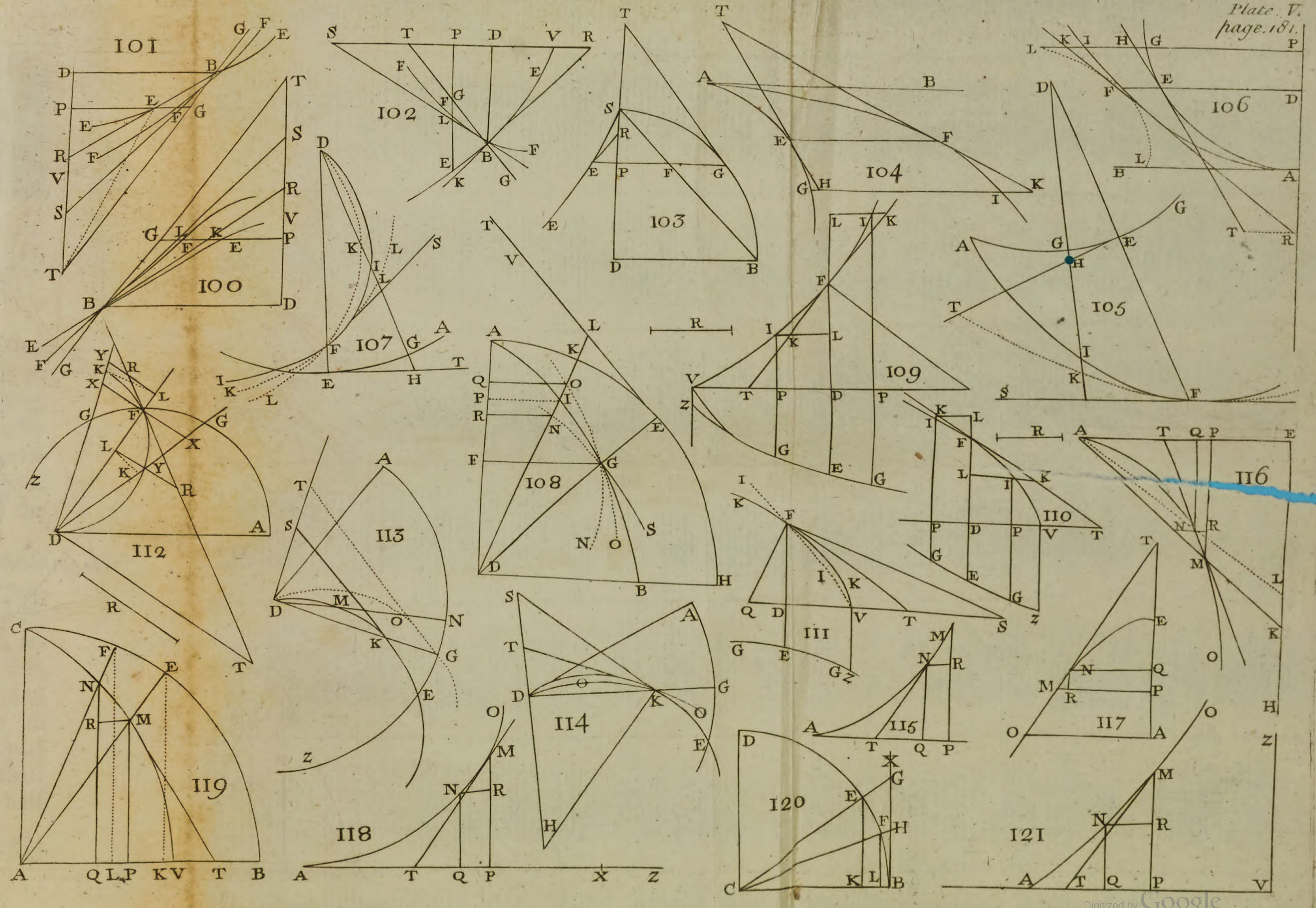
Let DEB be a Quadrant of a Circle, and BX a Tangent to it; and let the Line AMO be ſuch, that taking AP any how in AV, equal to the Arch BE, and erecting PM perpendicular to AV; let PM be equal to the Tangent BG of the Arch BE. Fig. 120. 121.

Aſſume the Arch BF = AQ, and drawing CFH, as alſo the Perpendiculars EK, FL to CB, call CB, r; CK, f; KE, g. Then becauſe it is CE : EK :: Arch EF : LK, or CE : EK :: QF : LK, that is, $r : g :: e : \frac{eg}{r} =$ LK; it ſhall be CL $= f+\frac{ge}{r}$. And LF $=\sqrt{rr-ff-\frac{2fge}{r}} = \sqrt{gg-\frac{2fge}{r}}$. But it is CL : LF :: (CB : BH ::) CB : QN. That is, $f+\frac{ge}{r} : \sqrt{gg-\frac{2fge}{r}} :: r : m-a$. Or (by ſquaring) $ff+\frac{2fge}{r} : gg-\frac{2fge}{r}$

$:: rr : mm - 2ma$. Whence (rejecting as neceſſary) we obtain the Equation $rfma = grre + gmme$. Therefore (by Subſtitution) $rfmm = grrt + gmmt$. Or $\frac{rfmm}{grr + gmm} = t$. Or (ſince m is $= \frac{rg}{f}$) it ſhall be $t = \frac{rr}{rr + mm} m$ $= \frac{CBq}{CGq} BG = \frac{CKq}{CEq} BG$.

Theſe Examples are ſufficient for the Illuſtration of our Method.

Plate V.
page. 181.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LECTURE XI.

THE former Part of our Buſineſs being in ſome ſort finiſhed, I ſhall now ſubjoyn ſome *Theorems* that offer relating to the *Menſuration* of *Magnitudes* by *Tangents* or *Perpendiculars* to *Curves*. Some of the Principal of which are theſe.

1. Let there be any Curve VH, whoſe Axis is VD, and Ordinate HD perpendicular to VD. Alſo let the Line *p*Z*y* be ſuch, that if from any Point, as E, taken at Pleaſure in it, the right Line EP, be drawn perpendicular to the Curve, and the right Line EAZ perpendicular to the Axis, let the right Line AZ be equal to the intercepted Part AP: then ſhall the Space AD*yp* be equal to one half the Square of the right Line DH. Fig. 122.

For let HDO be half a right Angle, and equally dividing the right Line VD indefinitely in the Points A, B, C; thro' theſe draw the right Lines EAZ, FBZ, GCZ parallel to the right Line HD, meeting the Curve in the Points E, F, G; from which draw the right Lines EIY, FKY, GLY parallel to VD or HO; as alſo let the right Lines EP, FP, GP, HP be perpendicular to the Curve. And let the Lines interſect one another as you ſee. The Triangle HLG is ſimilar to the Triangle PDH. For (becauſe of the indefinite Section or Diviſion) the ſmall Part GH of the Curve may be taken for a right Line. Therefore $HL : LG :: PD : DH$. And ſo $HL \times DH = LG \times PD$. That is, $HL \times HO = DC \times Dy$. In like manner, becauſe the Triangle GMF is ſimilar to the Triangle PCG, it will be $LK \times LY = CB \times CZ$; and alſo $KI \times KY = BA \times BZ$: And moreover $ID \times IY = AV \times AZ$. Therefore it is manifeſt that the Triangle HDO (not at all differing from the Rectangles $HL \times HO + LK \times LY + KI \times KY + ID \times IY$) is equal to the Space $VDyp$. Which does not differ from the Rectangles $DC \times Dy + CB \times CZ + BA \times BZ + AV \times AZ$. That is, $\frac{DHq}{2} =$ Space $VDyp$.

This may be demonſtrated indirectly or *ad abſurdum*, by a much longer Proceſs. But for what Reaſons?

2. Things being the ſame, as to Suppoſition and Conſtruction; the Sum of the Rectangles AZ × AE + BZ × BF + CZ × CG, *&c.* is equal to *three Times* the Cube of the Baſe DH. Fig. 122.

For becauſe HL : LG :: PD : DH :: PD × DH : DH*q*; it ſhall be HL × DH*q* = LG × PD × DH. That is, HL × HO*q* = DC × D*y* × DH. In like manner, LK × LY*q* = CB × CZ × CG And KI × KY*q* = BA × BZ × BF, *&c.* But HL × HO*q* + LK × LY*q* + KI × KY*q*, *&c.* is equal to three times the Cube of DH. Therefore the thing propoſed is manifeſt.

3. By the like Reaſon it is evident that the Sum of AZ × AE*q* + BZ × BF*q* + CZ × CG*q*, *&c.* is = $\frac{DHqq}{4}$; the Sum of AZ × AE Cub. + BZ × BE Cub. + CZ × CG Cub. *&c.* = $\frac{DH^5}{5}$. And ſo on.

Fig. 122. From whence by way of Conſectary we get the following (not contemptible) *Theorem.* Let VD*yp* be any ſpace, whoſe Axis VD is equally divided as aforeſaid. If each of the Spaces VAZ*p*, VBZ*p*, VCZ*p*, *&c.* be reſpectively drawn into

its Ordinates AZ, BZ, CZ, &c. The Sum ariſing will be equal to half the Square of the Space $VDyp$.

For (as has been demonſtrated already) the Space VDH may be adapted to the Figure $VDyp$ in ſuch manner, that drawing any right Line EP perpendicular to the Curve VH; AP may be equal to the correſpondent Ordinate AZ; and ſo

Lect. 10. (b) 1. of this. (b) the Space $VAZp = \frac{AEq}{2}$, and $VBZp = \frac{BFq}{2}$, and $VCZp = \frac{CGq}{2}$, &c. Wherefore all $VAZp \times AZ + VBZp \times BZ + VCZp \times CZ$, &c. are equal to all

$$\frac{AEq \times AZ + BFq \times BZ + CGq \times CZ}{2}$$

(c) 3. of this. that is, (c) $= \frac{DHqq}{4 \times 2}$: or (c) $= \frac{VDyp \times VDyp}{2}$.

5. If all $\sqrt{VAZp}$, $\sqrt{VBZp}$, $\sqrt{VCZp}$, &c. be drawn into their Ordinates AZ, BZ, CZ, &c. reſpectively, the Aggregate ariſing will be equal to two thirds of the ſquare Root of the Space $VDyp$ Cubed, *viz.* to $\frac{2}{3}\sqrt{VDyp}^{3}$.

For the Curve VH being adapted, it is $\sqrt{VAZp} = AE\sqrt{\frac{1}{2}}$; and $\sqrt{VBZp} = BF\sqrt{\frac{1}{2}}$, and $\sqrt{VCZp} = CG$, &c. And ſo ſince all $AZ \times AE + BZ \times BF + CZ \times CG$, &c. $= \frac{DH}{3}$ Cub. we ſhall have all $AZ \times \sqrt{VAZp} + BZ \times \sqrt{VBZp} + CZ \times \sqrt{VCZp}$, &c. $= \frac{DH}{3}$ Cub. $\times \sqrt{\frac{1}{2}} = \sqrt{\frac{DH^6}{18}}$. But $DHq = 2\,VDyp$, or $DH^6 = 8\,\overline{VDyp}^3$. Wherefore all $AZ \times \sqrt{VAZp} + BZ \times \sqrt{VBZp} + CZ \times \sqrt{VCZp}$, &c. $= \sqrt{\frac{8}{18}}\,\overline{VDyp}^3 = \frac{2}{3}\sqrt{\overline{VDyp}^3}$.

6. Let VD*y* be a Quadrant of a Circle, whoſe Radius call R, and Periphery P; the Segments VAZ, VBZ, VCZ, &c. drawn into the right Lines AZ, BZ, CZ, &c. will be equal to $\frac{RqPq}{8}$. Fig. 123.

Alſo the Sum $AZ\sqrt{VAZ} + BZ\sqrt{VBZ} + CZ\sqrt{VCZ}$ &c. $= \frac{2}{3}\sqrt{\frac{R^3P^3}{8}} = \sqrt{\frac{R^3P^3}{18}}$.

If V D*y* be a Segment of a *Parabola*; the Product of the Segments into the Ordinates ſhall be $\frac{2}{9}\,\mathrm{VD}q \times \mathrm{A}yq$; and that of the Roots of the Segments into the Ordinates will be $\frac{2}{3}\sqrt{\frac{8}{27}}\,\overline{\mathrm{VD}}^{3} \times \overline{\mathrm{D}y}^{3}$ $\surd\frac{32}{243}\,\overline{\mathrm{VD}}^{3} \times \overline{\mathrm{D}y}^{3}$.

From hence may be gotten ſeveral the like things, concerning the Products of the Powers or Roots of Segments drawn into the Ordinates or right Lines.

7. It follows too from what has been ſaid, if the ſeveral Abſciſſes V P be made Ordinates A Y, B Y, C Y, &c. at the Points A, B, C, &c. equal to them; the Space A D *x t* ſhall be equal to half the Square of the Subtenſe V H.

For becauſe $\mathrm{VA} + \mathrm{VB} + \mathrm{VC}$, &c. $= \frac{\mathrm{VD}q}{2}$, and all $\mathrm{AP} + \mathrm{BP} + \mathrm{CP}$, &c. $= \frac{\mathrm{DH}q}{2}$. it is evident that all the V P's $= \frac{\mathrm{VH}q}{2}$.

8. Again, (ſuppoſing the ſame things) Let the Curve R X X S be ſuch, that if $\mathrm{IX} = \mathrm{AP}$, and $\mathrm{KX} = \mathrm{BP}$, and $\mathrm{LX} = \mathrm{CP}$, &c. then the *Solid* generated by the Rotation of the *Space* V D *y p* about the Axis V D will be one half the *Solid* generated

by the Rotation of the *Space* DRSH about the Axis VD.

For becauſe $HL : LG :: PD : DH :: Dy : DH :: Dyq : Dy \times DH :: Dyq : HS \times DH$; it will be $HL \times HS \times DH = LG \times Dyq = DC \times Dyq$. By the very ſame Way of Reaſoning it ſhall be $LK \times LX \times DL = CB \times CZq$; and $KI \times KX \times DK = BA \times BZq$, &c. But the former Solid is $\frac{d}{p} : AZq + BZq + CZq$, &c. And the latter Solid is $\frac{2d}{p} : DI \times IK + DK \times KX + DL \times LX$, &c. And conſequently the thing propoſed is manifeſt.

Note, $\frac{d}{p}$ *is the Ratio of the Diameter of a Circle to the Circumference.*

9. All theſe *Theorems* take place and are demonſtrated after the ſame manner, when the Curve VEH is Convex towards the right Line VD. As, for Example, taking any Point E in the Curve, and drawing thro' it the right Line EP, perpendicular to the Curve VEH, and EAY perpendicular to the right Line VD, and making $AZ = AP$; the Space VDy ſhall be $= \frac{DHq}{2}$. And if you make $AY = VP$; the Space VDx will be $= \frac{VDq}{2}$. And the like of the reſt. Fig. 124.

Experience will eaſily ſhew what a vaſt Number of *Magnitudes* do receive their *Dimenſions* from the immediate Conſtruction of theſe *Theorems.*

Fig. 125. 10. Again, let there be any Curve VH, whoſe Axis is AD, and Baſe DH, and the Line DZZO ſuch, that drawing the right Line ET from any Point E in the Curve to touch the ſame, and the right Line EIZ parallel to the Baſe, we ſhall have always IZ equal to AT. I ſay the *Space* DHO is equal to the *Space* VDH.

For let the right Line DH be equally but indefinitely divided, in the Points I, K, L, thro' which draw the right Lines EIZ, FKZ, GLZ, parallel to VD meeting the Curve in the Points E, F, G, from whence draw the right Lines EA, FB, GC parallel to HD, and the right Lines ET, FT, GT, as alſo HT *touching the Curve.* And let the Lines interſect one another as *per* Scheme. Now the Triangle GLH is ſimilar to the Triangle TDH, for by Reaſon of the ſaid indefinite Diviſion, the ſmall Arch GH may be looked upon as a right Line, in as much as it coincides with the Tangent. Therefore $LG : LH :: TD : DH$. And $LG \times DH = LH \times TD$, or $CD \times DH = LH \times HO$. In like manner $BC \times CG = KL \times LZ$, and $AB \times BF = IK \times KZ$, and VA

$\times AE = DI \times IZ$. But the Sum $CD \times DH + BC \times CG + AB \times BF + VA \times AE$ does not differ from the Space VDH, and the Sum $LH \times DO + KL \times LZ + IK \times KZ + DI \times IZ$ does not differ from the Space DHO; and ſo the Space VDH is equal to the Space DHO.

We are indebted to the learned Dr. *Gregory* of *Aberdeen*, for this uſeful *Theorem.* To which we ſubjoyn theſe that follow.

11. The ſame things being ſuppoſed, the Solid generated by the Rotation of the Space DHO about the Axis VDR is twice the Solid generated by the Rotation of the Space VDH about the Axis VD. Fig. 125.

For it is $HL : LG :: (DH : DT :: DH : HO ::) DHq : DH \times HO$. Whence $HL \times DH \times HO = LG \times DHq = CD \times DHq$. In like manner $LK \times DL \times LZ = BC \times CGq$ and $KI \times DK \times KZ = AB \times BFq$. And laſtly, $ID \times DI \times IZ = VA \times AEq$. But it is well known that the Sum of $CD \times DHq + BC \times CGq + AB \times BFq + VA \times AEq$, &c. is twice the Sum of $DI \times IE + DK \times KF + DL \times LG$, &c. Therefore the Solid generated by the Rotation of the Space HDO about the Axis DR is twice the Solid generated by the Rotation of the Space VDH about the Axis VD.

12. Hence the Sum of $DI \times IZ + DK \times KZ + DL \times LZ$, &c. is equal to the Sum of the Squares of the Ordinates to VD, *viz.* equal to $AEq + BFq + CGq$, &c.

13. By a like Reaſoning it is evident that the Sum of $DIq \times IZ + DKq \times KZ + DLq \times LZ$, &c. is three times the Sum of $DIq \times IE + DKq \times KF + DLq \times LG$, &c. that is, equal to the Sum of the Cubes of the Ordinates AE, BF, CG, &c. to VD. The Tenor of the Concluſions is obſerveable as to the reſt of the *Powers.*

14. The ſame things ſuppoſed, let the Curve DXH be ſuch, that any Ordinate, as IX to DH is a mean Proportional between the correſpondent Ordinates IE, IZ; the Solid generated by the Rotation of the Space VDH about the Axis DH is twice the Solid ariſing from the Generation of the Space DXH about the ſame Axis DH.

For becauſe $VA \times AE = DI \times IZ$; it ſhall be $VA \times AE \times EI = DI \times IZ \times IE = ID \times IXq$. And by the ſame Reaſon $AB \times BF \times FK = IK \times KXq$, and $BC \times CG \times GL = KL \times LXq$, &c. But the Sum of $VA \times AE \times FI + AB \times BF \times FK + BC \times CG \times GL$, &c. is one half of the Sum of $VDq + EIq + FKq + GLq$. Therefore

In 10. of this.

fore the Sum of $IXq + KXq + LXq + HXq$ is one half the Sum of $VDq + EIq + FKq + GLq$. Therefore what was propoſed is manifeſt.

15. If the Curve DXH be ſuch, that any Ordinate, as IX, be a *Bimedial* between the congruous Ordinates IE, IZ; the Sum of the Cubes of IX, KX, LX, *&c.* will be one third Part of the Cubes of DV, IE, KF, *&c.* If IX be a *Trimedial*, the Sum of $IXqq + KXqq + LXqq$, *&c.* will be $= \frac{DVqq + IEqq + KFqq}{4}$, *&c.* And ſo on with regard to other Powers.

Note, I call a *Bimedial*, the firſt of two mean Proportionals; a *Trimedial*, the firſt of three, *&c.*

This is gathered and demonſtrated after the ſame Manner. Repetition is irkſome.

16. Again let the Line VYQ be ſuch that the Ordinate AY be equal to AT, and the Ordinate BY, *&c.* be equal to BT; then ſhall $IZq + KZq + LZq$, *&c.* (the Sum of the Squares of the Ordinates of the Curve DZO to the right Line DH) be equal to the Sum of $VA \times AE \times AY + AB \times BF \times BY + BC \times CG \times CY$, *&c.* That is, equal to the Figure VDH drawn into the Figure VDQ.

17. Alſo, the Sum of $IZ\ \text{Cub.} + KZ\ \text{Cub.} + LZ\ \text{Cub.}$ &c. $= VA \times AE \times AYq + AB \times BE \times BYq + BC \times CG \times CYq$, &c. that is, equal to the Figure VDH drawn into the Squares of the Figure VDQ.

The like Reaſon holds of other Powers too.

Theſe may be eaſily brought to the ſame Law with thoſe above.

Fig. 126. 18. All this is true, and demonſtrated in like ſort, when the Curve VH is Convex towards the right Line VD: For Example, if the Line DZO be ſuch, that drawing thro' any Point E, in the Curve the Tangent ET, and EA parallel to HD, and EIZ parallel to VD, we have always $IZ = AT$: The Space DHO ſhall be equal to the Space VDH, and the Solid generated by the Rotation of the Space DHO about the Axis VR will be twice the Solid generated by the Rotation of the Space VDH about the ſame Axis VD. And ſo of the reſt.

Fig. 127. 19. Again, let there be any Curve AMB, whoſe Axis is AD, and BD perpendicular to it; and let the Line KZL be ſuch, that taking the Point M any where in AB, and drawing thro' this the right Line MT touching the Curve AB, the right Line MFZ parallel to DB (cutting the Line KL in Z, and the right Line AD in F) and taking any right Line R, it may be $TF : FM :: R : FZ$. Then will the Space $ADLK$ be equal to the Rectangle under R and DB.

For let DH be $= R$, and compleat the Rectangle $BDHI$. And taking MN an indefinitely ſmall Part of the Curve AB, draw NG parallel to BD, and MEX, NOS parallel to AD. Then is $NO : MO :: TF : FM :: R : FZ$. Whence $NO \times FZ = MO \times R$, that is, $FG \times FZ = ES \times EX$. Therefore ſince all the Rectangles $FG \times FZ$ do not differ from the Space $ADLK$, and the ſame Number of Rectangles $ES \times EX$ do make up the Rectangle $DHIB$. The thing propoſed is ſufficiently manifeſt.

20. The ſame things ſuppoſed, if the Curve PYQ be ſuch, that taking the Ordinate EY in the aſſumed right Line MX equal reſpectively to FZ: The Sum of the Squares of FZ (computed on the right Line AD) will be equal to the *Space* $DBQB$ drawn into R.

For it is $FG : ES :: NO : MO :: R \times FZ : FZq :: R \times EY : FZq$. And ſo $FG \times FZq = ES \times R \times EY$.

21. In like manner, the Sum of the *Cubes* of FZ is equal to the Sum of the Squares of the right Lines EY (apply'd to BD) drawn into R. And a like Law

holds with regard to the reſt of the Powers.

Fig. 128. 22. Let there be any Curve DOK, in which the Point D is taken at pleaſure, and ſuppoſe DK to be a Subtenſe. Alſo let the Curve AE be ſuch, that drawing any right Line DMF from D, cutting the Curves in the Points M, F, and drawing DS perpendicular to DM, and MS touching the Curve DOK, meeting in the Point S, and let R be ſome given right Line, I ſay when this is done, let DS : 2 R :: DM*q* : DF*q*. Then ſhall the Space ADE be equal to the Rectangle under R and DK.

For let the Subtenſe DK be conceived to be indefinitely divided in the Points P, Q, &c. thro' which let the Arches PM, QRN deſcribed from the Centre D paſs, cutting the Curve DOK in the Points M, N; thro' which draw the right Lines DMF, DNG. Alſo let DT be perpendicular to DK, and DS perpendicular to DM, meeting the Tangents KT, MS. Again from the Centre D deſcribe the Arch EX thro' the Point E, and the Arch FY thro' F. Now becauſe of the indefinite Section, the Triangle KPM is ſimilar to the Triangle KDT. And ſo MP : PK :: TD : DK. Alſo it is DP : PM :: DE : EX. Or for the ſaid reaſon, DK : MP :: DE : EX. And ſo MP × DK :

$PK \times MP :: TD \times DE : DK \times EX$. That is, $DK : PK :: TD \times DEq : DK \times EX \times DE$. Therefore $DKq \times EX \times DE = PK \times TD \times DEq$. (a) But $DT : 2R :: DKq : DEq$, or $DT \times DEq = 2R \times DKq$. Wherefore $DKq \times EX \times DE = PK \times 2R \times DKq$. And ſo $EX \times DE = 2R \times PK$. That is, 2 Sector $DEX = 2R \times PK$. Whence the Sector $DEX = R \times PK$. By the ſame Way of Reaſoning the Sector DFY is equal to $R \times RM$, or $R \times QP$. And ſo the whole Space ADE made up of theſe Sectors is equal to the whole Rectangle $R \times DK$. Which was propoſed.

(a) Supp.

23. The ſame Suppoſition and Conſtruction remaining, draw KH perpendicular to KT, and MI to MS, and conceive the Curve AE now to be of ſuch a Nature, that DE be $= \sqrt{DK \times DH}$, and $DF = \sqrt{DM \times DI}$, and ſo on continually: The Space ADE will be one fourth of the Square of DK.

Fig. 128.

For it is $MP : PK :: DK : DH :: DKq : DK \times DH :: DKq : DEq$. Alſo it is $DP : PM :: DE : EX$. That is, $DK : PM :: DE : EX$. Therefore $MP \times DK : PK \times PM :: DKq \times DE : DEq \times EX$. That is, $DK : PK :: DKq : DE \times EX$. Or $DKq : DK \times PK :: DKq : DE \times EX$. Whence $DK \times PK = DE \times EX$. In like manner DM × MR (or DP × PQ) = DF × FY. But all the Rectangles DK × PK, DP × PQ, &c. are equal to one half the Square of DK, and all the Rectangles DE × EX, DF × FY, &c. are equal to twice the Space EDA. Wherefore the Propoſition manifeſtly follows.

Fig. 129 24. Let there be any Curve DOK, and the Point D in the ſame, and let DK be a Subtenſe thereof; alſo let the Curve DZI be ſuch, that taking any Point M in it, joyning DM, drawing DS perpendicular to DM, and MS to touch the Curve DOK, and taking DP = DM, and and drawing PZ perpendicular to DK; let PZ = DS. Then will the Space DKI be twice the Space DKOD.

For conceive the right Line KP to be indefinitely ſmall, and let DT be perpendicular to DK, and KT touch the Curve DOK. And ſo (drawing the Arch MP) it is again KP : PM :: KD : DT :: KD : KI. Whence KP × KI = PM × KD. Take another Particle PQ and from the Centre D thro' Q draw the Arch QN cutting the Subtenſe DM in R. Then too we ſhall have MR : RN :: MD : DS. That is, PQ : RN :: MD : PZ. Wherefore PQ × PZ = RN × MD; and ſo on continually. Therefore it is manifeſt that all the Rectangles together KP × KI, PQ × PZ, &c. are equal to the aggregate of all the Rectangles $PM \times KD$, $RN \times MD$, &c. That is, the Space DKI is twice the Space DKOD.

25. The ſame Suppoſition and Conſtruction remaining, let the Ordinates PZ be now conceived reſpectively equal to MS, and to an aſſum'd right Line X*k*, and at the Diſtances X*k*, X*m*, X*n*, &c. equal to the Parts DOK, DOM, DON, &c. of the Curve; let the right Lines *kd*, *md*, *nd*, &c. be apply'd equal to the Subtenſes KD, MD, ND, &c. Then will the Space X*kd* be equal to the Space DKI. Fig. 130.

For it is $KM : KP :: KT : KD$. That is $km : KP :: KI : kd$. Whence $km \times kd = KP \times KI$. In like manner, $MN : MR :: MS : MD$. Or $mn : PQ :: PZ : md$. Whence $mn \times md = PQ \times PZ$, and ſo on. Whence the Propoſition is manifeſt.

26. Moreover, things remaining as before, if any right Line *kg* be taken, the Rectangle X*kgh* compleated, and the Curve DZI be ſuch, that $MD : MS :: kg : PZ$: the Rectangle X*kgh* ſhall be equal to the Space DKI. Fig. 130.

For it is again $KP : KM :: DD : KT :: kg : KI$. And ſo $KP \times KI = (KM \times kg =)\ km \times kg$. In like manner $PQ \times PZ = mn \times kg$. And ſo on. Therefore the Propoſition is manifeſt.

Hence the Space D K I being known; the Quantity of the Curve D O K will be known too.

Any one who has leiſure and inclination, to make a narrower Search, and extend this *Mine*, will diſcover many other Concluſions of this kind.

27. The following Theorem deduced too from the Premiſſes will perhaps be of ſome Uſe; it has been ſo to me oftentimes.

Fig. 131. Let there be any Curve V E H, whoſe Axis is V D, and Baſe D H, which the right Line E T, any how touches, and draw E A parallel to H D. Conceive another Curve G Z Z to be of ſuch a Nature, that drawing a right Line E Z from the Point E parallel to V D, cutting the Baſe D H in the Point I, and the Curve G Z Z in the Point Z, and taking any determinate Line R, let it be always D A *q* : R *q* :: D T : I Z. Then ſhall it be D A : A E :: R *q* : Space D I Z G. (Or making D A : R :: R : D P, and drawing P Q parallel to D H) The *Rectangle* D P Q I will be equal to the *Space* D G Z I.

Let this *Theorem* be ſubjoyn'd; of ſome Uſe hereafter.

Fig. 132. 28. Let there be any Curve A M B, whoſe Axis is A D. Alſo let the Line K Z L have ſuch a Nature, that taking any Point M in A M B, and drawing from it the right Line M P perpendicular to the Curve A B, cutting the Axis A D in the Point P, and the right Line M G perpendicular to A D, interſecting the Curve K Z L in the Point Z, it may be always G M : M P :: Arch A M : G Z. Then ſhall the Space A D L K be equal to half the Square of the Arch A M.

There is no great Difficulty in deducing theſe *Theorems* from thoſe that go before, which Admonition let be ſufficient: For I have now a Mind to reſt a while.

APPENDIX.

1. HAVING ſome Years ago fully conſider'd the *Cyclometry* of the Celebrated Mr. *Huygens*, I obſerv'd that he uſes two Ways more eſpecially for that purpoſe. The *Firſt* is, his ſhewing the Segment of a Circle to be a Mean between two parabolick Segments; the one inſcribing, and the other Circumſcribing, and thence preſcribes the Limits of it. And the *other*, his demonſtrating the Centre of Gravity of the Segment of a Circle lying between the Centre of Gravity of a *Parallelogram* and Segment of a *Parabola* of the ſame Height, does thereby aſſign other Limits; a Thought came into my Head, that inſtead of a *Parabola* in the firſt Method, and a *Parallelogram* in the ſecond, ſome

one of the *Paraboliform* Figures, circumſcribing the Segment of a Circle might be uſed, and ſo the Buſineſs obtain'd ſomething nearer; preſently upon Reflection I found the Thing was true, and obſerv'd withal that the like Methods might eaſily be accommodated to the *Menſuration* of the *Segment* of an *Hyperbola.* The following Demonſtration of which, being brief and clear (in reſpect of others, perhaps, that may be deviſed) and withal flowing from or depending upon what foregoes, I have thought fit to lay down here.

2. For this purpoſe, we muſt aſſume these common *Theorems* whoſe Demonſtrations are not difficult to be deduced ſeveral Ways from what has been already eſtabliſhed. If $\frac{n}{m}$ be the Exponent of a *paraboliform* Figure BAE, whoſe Axis is AD, Baſe or Ordinate BDE, Tangent BT, and Centre of Gravity K. Then ſhall the Area BAE $= \frac{m}{n+m}$ AD × BE, and TD $= \frac{m}{n}$ AD, and KD $= \frac{m}{n+2m}$ AD. Fig. 133.

3. Let

Fig. 134. 3. Let there be any two Curves AEB, AFB (whoſe common Axis is AD, and Ordinate DB) ſo diſpoſed that drawing any right Line EFG parallel to BD, cutting the Lines (drawn) in the Points E, F, G, and ſuppoſing the right Lines ES, FT to touch the Curves, the former AEB, and the latter AFB; TG be perpetually greater than SG: I ſay no part of the Curve AFB will fall within AEB.

If poſſible, let the Part NFM fall; *viz.* ſo that the Curve AFB interſects the Curve AEB in the Points M, N, between which conceive the indeterminate Ordinate EFG to fall. And let the Lines LXK, RYQ be ſuch, that drawing the right Lines EO, FP perpendicular to ES, FT, and continuing out the right Line EG, cutting the ſaid Lines LK, QR in the Points X, Y; GX be = GO, and GY = GP. Now from what has been demonſtrated, it is manifeſt that the Space IHKL = $\frac{HMq - INq}{2}$ = Space IHQR; and ſo the Space IHKL is equal to the Space IHQR. But becauſe GE : GO (GX) :: SG : GE $\sqsupset$ SG : GF $\sqsupset$ TG : GF :: GF : GP (GY) $\sqsupset$ GE : GY; it is GX $\sqsubset$ GY. Conſequently becauſe this is always ſo, the Space IHKL is greater than the Space IHQR. Which is contrary to what has been demonſtrated. Therefore the Propoſition is evident. Hence the whole Curve AFB falls entirely without the whole Curve AEB, neither can that ever Interſect this.

4. Let there be any Curve BAE, whoſe Axis is AD and let BDE be an Ordinate or Baſe to the ſame. Alſo let the Point H be the Centre of Gravity of the Segment BAE, thro' which let the right Line RS be drawn parallel to BE. Moreover thro' the Points RS let ſome other Line MRASN paſs, having the ſame Axis AD, and interſecting the former Curve BAE in ſuch manner, that its upper Part RKAPS falls within the Curve BAE, and the lower Parts RM, SN (remaining) without the ſame. Then will the Centre of Gravity of the Segment MRSAN fall below the Point H towards the Baſe MN. Fig. 135.

For when RIAK + AOSP is taken away from the Segment RIAOS, the Remainder BRKAPSE falls down towards the Baſe BE; ſuppoſe now that the Centre of Gravity of this Remainder be at X; then adding BRM + ESN it will ſtill cauſe the whole MRKAPSN more to decline, and ſo its Centre will fall low X, ſuppoſe be at Y. Therefore the Propoſition is manifeſt.

5. Let

Fig. 136. 5. Let two right Lines B T, E S touch the Circle A F B, whoſe Centre is C, meeting the Diameter C A in the Points T, S; and let the right Lines B D, E P be perpendicular to C A; and let A D be greater than A P. Then will T D : A D ⊏ S P : A P.

For it is, C T : C A :: C A : C D. And ſo C T — C A : C A — C D :: C T : C A. That is, T A : A D : C T :: C A. In like manner it is manifeſt that S A : A P :: C S : C A. But C T : C A ⊏ C S : C A. Wherefore T A : A D ⊏ S A : A P. Or *(Componendo)* T D : A D ⊏ S P : A P.

Fig. 137. 6. Let two right Lines B T, E S touch the *Hyperbola* A E B, whoſe Centre is C, and ſuppoſe all things elſe, as in our laſt, it will be T D : A D ⊐ S P : A P.

For it is C A : C D :: C T : C A. Whence C A — C T : C D — C A :: C T : C A. That is, T A : A D :: C T : C A. In like manner it is S A : A P :: C S : C A. But C T : C A ⊐ C S : C A. Wherefore T A : A D ⊐ S A : A P. Or *(Componendo)* T D : A D ⊐ S P : A P.

Fig. 138. 7. Let the Circle A E B, whoſe Centre is C, and *paraboliform* Figure A F B have the ſame Axis D A D, and Baſe B D, and let the Exponent of the *paraboliform* Figure be $\frac{n}{m}$, and AD $= \frac{m - 2n}{m - n}$ C A (or

$m - n : m - 2n$ :: CA : AD) and the right Line BT touch the Circle: This will alſo touch the *paraboliform* Figure AFB.

For becauſe BD touches the Circle, it is CT : CA :: CA : CD. Whence TA : AD :: CA : CD. And *(Componendo)* TD : AD :: CA + CD : CD. Alſo becauſe (by *Suppoſition)* CA : AD :: $m - n : m - 2n$. It will be (by *Inverſion)* CA : CD :: $m - n : n$. And *(Componendo)* CA + CD : CD :: $m : n$. That is, TD : AD :: $m : n$. Whence *(a)* it is evident that BT touches the *paraboliform* Figure AFB.

8. We may obſerve that this *Theorem* may be inverted, *viz.* the Ratio of AD to CA being given, the *paraboliform* Figure may from thence be determined, that will touch the Circle AEB in the Point B. For if AD $= \frac{s}{t}$ CA; then will $\frac{t - s}{2t - s}$ be the Exponent of the ſaid *paraboliform* Figure. For let $\frac{t - s}{2t - s}$ be $= \frac{n}{m}$; and (multiplying croſs-wiſe) it will be $mt - ms = 2tn - sn$, and by Tranſpoſition $mt - 2nt = ms - ns$. And ſo reducing the Equality to an Analogy, $m - n : m - 2n :: t : s$ :: CA : AD. And therefore the Thing is evident by our laſt.

(a) 2 *of this Appen.*

9. The Suppoſition of N. 7. remaining, the *paraboliform* Figure A F B will fall entirely without the Circle A E B.

For draw the right Line G E F any how parallel to D B, cutting the Circle in E, and the *paraboliform* Figure in F, and conceive the right Line E S to touch the Circle, and the right Line F R the *Paraboloid.* Then it is : R G : A G :: (*a*) *m* : *n* :: T D : A D (*b*) $\sqsubset$ S G : A G. Wherefore R G $\sqsubset$ S G. And conſequently (*c*) it follows that the whole *Paraboloid* A F B falls without the Circle.

(*a*) 2. *of this Appen.*
(*b*) 5. *of this Appen.*
(*c*) 3. *of this Appen.*

Fig. 139.

10. Things remaining as they were. if a *Paraboloid* of the ſame kind with A F B, *viz.* whoſe Exponent is $\frac{n}{m}$, be ſuppoſed to ſtand upon the Baſe G E any how parallel to D B, and having A D for its Axis. This will fall above G E towards A quite without the Circle.

For taking any Point M in the Arch A E, and drawing M P parallel to E G, and M V to touch the Circle. We have V P : A P $\sqsupset$ S G : A G $\sqsupset$ R G : A G :: *m* : *n*. Therefore (*a*) the Propoſition is again manifeſt.

(*a*) 2. *of this Appen.*

Fig. 139.

11. It follows too that the ſaid *Paraboloid* (alike to A F B and ſtanding upon the Baſe G E) continu'd out below G E towards D B will intirely fall within the Circle.

That it will fall below E within the Circle, follows becauſe R E touching it cuts the Circle, ſince S E touches the Circle. That it will no where meet the Circle is evident from hence; if it be ſuppoſed to meet it any where as at N, it will *(a)* entirely fall without the Circle above N. Which is contrary to what has been but now aſſerted and demonſtrated.

(a) 3. *of this Appen.*

12. Again, let the *Hyperbola* A E B, whoſe Centre is C, and the *Paraboloid* A F B, whoſe Exponent is $\frac{n}{m}$ have the ſame Axis A D and Baſe D B. Alſo let A D $= \frac{2n - m}{m - n}$ C A, and B T touch the *Hyperbola.* This will alſo touch the *Paraboloid* A F B.

Fig. 140.

For it is C D : C A :: C A : C T. And ſo A D : T A :: C D : C A. And by (Inverſion and Compoſition T D : A D :: C A + C D : C D. But (by Suppoſition) it is $m - n : 2n - m$ :: C A : C D. And ſo (by Inverſion and Compoſition) C A : C D :: $m - n : n$. And again (by Compoſition) C A + C D : C D :: $m : n$. That is, T D : A D :: $m : n$. Wherefore B T touches the *Hyperbola.*

13. Hence again, when the Ratio of A D to C A is given, the *Paraboloid* touching the *Hyperbola* at the Point B may be determined. For let A D $= \frac{s}{t}$ C A; then ſhall $\frac{n}{m} = \frac{t + s}{2t + s}$. Now multiplying croſs-wiſe $2\,t\,n + s\,n = m\,t + m\,s$. And (tranſpoſing) $2\,n\,t - m\,t = m\,s - n\,s$. Therefore $m - n : 2\,n - m :: t : s ::$ C A : A D. Conſequently the Thing is evident by our laſt.

Fig. 141. 14. The Suppoſition of N. 12. remaining, the *Paraboloid* A F B will fall quite within the *Hyperbola* A E B.

For any how draw the right Line E F G parallel to BD, and let the right Line E R touch the *Hyperbola*, and the right Line F S the *Paraboloid*. Now it is S G : A G :: (*a*) $m : n ::$ T D : A D (*b*) $\sqsupset$ R G : A G. Whence R G $\sqsubset$ S G. And ſo (*c*) the Curve A E B will fall quite without the Curve A F B.

(*a*) 2. *of this Appen.*
(*b*) 6. *of this Appen.*
(*c*) 3. *of this Appen.*

Fig. 141. 15. Alſo, things remaining the ſame, let a *Paraboloid* of the ſame Order be conceived to have G E for its Baſe, and A G its Axis. This will fall quite within the *Hyperbola* above G E.

For if the Point M be taken any where in the Curve A E of the *Hyperbola*, and the Ordinate M P, and Tangent M V to the *Hyperbola* be drawn, it ſhall be $VP : AP \sqsubset m : n$, And ſo the thing propoſed is evident from *n.* 3.

Fig. 141. 16. Moreover, if a *Paraboliform* Figure of the ſame Order ſuppoſed to ſtand on the Baſe E G; be conceived to be continued out to D B; the Part thereof intercepted between E G, B D, will fall without the *Hyperbola.*

For that it falls without the *Hyperbola* below E G is evident, becauſe it makes a leſs Angle with the right Line E S the Tangent of it, than the ſame right Line E S makes with the right Line E R touching the *Hyperbola.* And that the ſame will no where meet the *Hyperbola*, is evident from hence, that ſuppoſing it to do ſo, as at N; it would fall *(a)* entirely within the *Hyperbola* A N. Which is contrary to what has but now been ſhewn.

(a) 3. of this *Appen.*

Fig. 142. 17. Let the Circle A E B and *Parabola* A F B have the ſame Axis A D, and Baſe D B. The *Parabola* will fall within the Circle above B D, and without it below B D.

For let A Z be the Diameter of the Circle, and A H parallel to B D equal to the ſame. Joyn Z H, and continue out B D meeting it in the Point I. Then is D I the *Parameter* of the *Parabola* A F B. Now if a right Line E F G K be any how drawn above BD parallel to BD cutting the Circle in E, the *Parabola* in F, and the right Lines AZ, HZ in the Points G, K, it is manifeſt that $GEq = AG \times GK \sqsubset AG \times DI = GFq$ Whence $GE \sqsubset GF$. Alſo if the right Line MNOL parallel to BD be any how drawn below BD, cutting the *Parabola* in M, the Circle in N, and the right Lines AZ, HZ in O and L, it is alſo evident $MOq = AO \times DI \sqsubset AO \times OL = NOq$. And ſo $MO \sqsubset NO$. Therefore the Propoſition is evident.

If inſtead of a Circle be put an Ellipſis, the ſame Concluſion follows from the ſame Reaſoning; ſuppoſing AH to be the *Parameter of the Ellipſis.*

Fig. 143. 18. Let the *Hyperbola* AEB, (whoſe Axis is AZ and *Parameter* AH) and the *Parabola* AFB have the ſame Axis AD and Baſe DB; the *Parabola* will fall quite without the *Hyperbola* above BD, and quite within it, when continued below BD.

For let BD continued out meet the Line joyning Z, H in the Point I; then DI is the *Parameter* of the *Parabola.* Now if the right Line FEGK be any how drawn above BD parallel to BD, interſecting the *Hyperbola* in E, the *Parabola* in F, and the right Lines AD, ZH in the Points G, K, it ſhall be $FGq =$

$AG \times DI \sqsubset AG \times GK = EGq$. Wherefore $FG \sqsubset EG$. So alſo if the right Line MNOL be any how drawn below BD cutting the *Hyperbola* in N, the *Parabola* in M, and the right Lines AD, ZH in the Points O, L, it ſhall be $NOq = AO \times OL \sqsubset AO \times DI = MOq$. And ſo $NO \sqsubset MO$. Therefore the Propoſition is manifeſt.

19. From what has been ſaid may be Fig. 144. gathered the following Rules relating to the *Menſuration* of a *Circle.* Let BAE be a Segment of a Circle, whoſe Axis is AD, and Baſe BE, and let C be the Centre of the Circle, and EH the right Sine of the Arch BAE, alſo let $AD = \frac{s}{t} CA$. Then it ſhall be

1, $\frac{2t - s}{3t - 2s} AD \times BE \sqsubset$ Segm. BAE.

2. $EH + \frac{4t - 2s}{3t - 2s} BH \sqsubset$ Arch BAE.

3 $\frac{2}{3} AD \times BE \sqsupset$ Segm. BAE.

4. $EH + \frac{4}{3} BH \sqsupset$ Arch BAE.

Fig. 145. 20. In like manner are deduced theſe Rules regarding the *Menſuration* of the *Hyperbola.* Let ADB be a Segment of an *Hyperbola*, whoſe Centre is C, Axis AD $= \frac{s}{t}$ CA, and Baſe DB. Then it ſhall be

1. $\frac{2t+s}{3t+2s}$ AD × DB $\sqsupset$ Segm. ADB.

2. $\frac{2}{3}$ AD × DB $\sqsubset$ Segm. ADB.

21. Again let BAE be the Segment of a Circle whoſe Centre is C, Axis AD, and Centre of Gravity K. And put AD $= \frac{s}{t}$ CA, and HD $= \frac{2t-s}{5t-3s}$ AD, then ſhall HD be greater than KD.

Fig. 146. For thro' H draw the right Line OP parallel to BE. But the Point H is *(a)* the Centre of Gravity of the *Paraboloid*, ſuppoſe AFB, ſtanding upon the Baſe BE, whoſe Exponent is $\frac{t-s}{2t-s}$, and accordingly will touch *(b)* the Circle AEB, (for if $\frac{t-s}{2t-s} = \frac{n}{m}$; $\frac{2t-s}{5t-3s}$ will be $= \frac{m}{n+2m}$) and ſo H will be *(a)* the Centre of Gravity of the *Paraboloid* of the

(a) 2. of this *Appen.*

(b) 8. of this *Appen.*

the ſame Kind paſſing thro' O P, and reaching to the Baſe BE; this above OP will fall *(c)* without the Circle, and below *(d)* O P within the Circle. Conſequently the Point H is *(e)* ſituate above the Point K.

(c) 10. of this *Appen.*
(d) 11. of this *Appen.*
(e) 4. of this *Appen.*

Fig. 146.

22. But if the Point L be the Centre of Gravity of the *Parabola*; L will fall below K; and ſo $KD \sqsubset \frac{2}{5} AD$. As is manifeſt from the 4th and 17th Numbers of this *Appendix*.

Fig. 147.

23. Let B A E be the Segment of an *Hyperbola*, whoſe Centre is C, Axis A D, Baſe B E, and Centre of Gravity K; alſo put $AD = \frac{s}{t} CA$, and $HD = \frac{2t+s}{5t+3s} AD$; then ſhall H D be leſs than K D.

For draw the right Line O P thro' H parallel to B E. But *(a)* the Point H is the Centre of Gravity of the *Paraboloid*, ſuppoſe A F B, ſtanding upon the Baſe D B, whoſe Exponent is $\frac{t+s}{2t+s}$; which will alſo touch *(f)* the *Hyperbola* at B (for if $\frac{t+s}{2t+s} = \frac{n}{m}$; $\frac{2t+s}{5t+3s}$ will be $= \frac{m}{n+2m}$.) Wherefore H will be the Centre of Gravity of a *Paraboloid*

(f) 13. of this *Appen.*

of the ſame Kind paſſing thro' O P, and extending to B E. But this falls within
(g) 15. of this *Appen.* the *Hyperbola* above O P, and without (g)
it below O P. Conſequently the Point K
(h) 16. of this *Appen.* falls (h) above H.

24. The Centre of Gravity ſuppoſe L, of the *Parabola*, falls above the Point K; and ſo K D $\sqsupset \frac{2}{5}$ A D. As is manifeſt from the 4th and 18th Numbers of this *Appendix*.

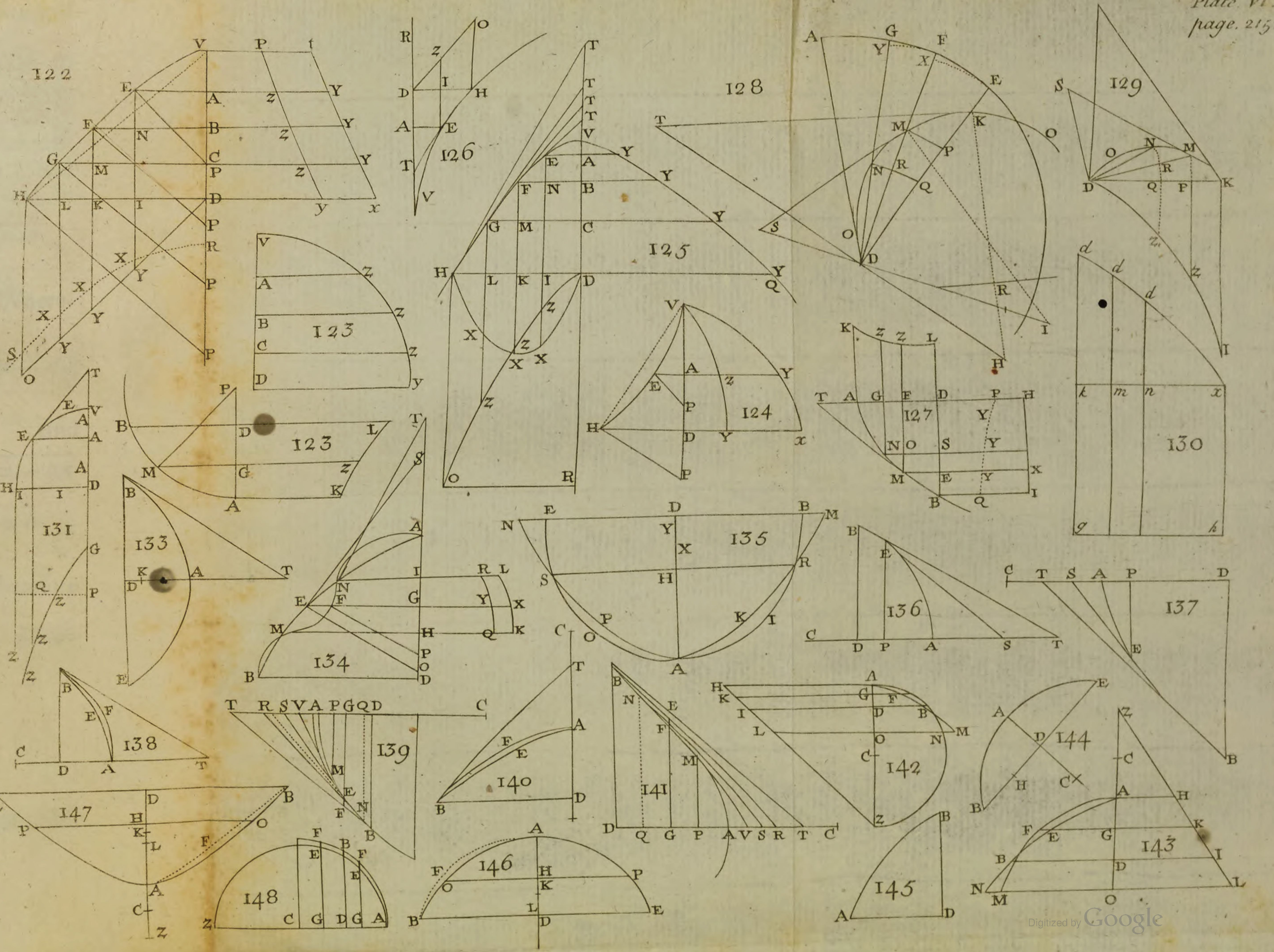
25. Leſt the preſent Speculations may ſeem of no great Value, becauſe of the daily Publication of ſeveral of theſe Sort of *Cyclometrical Methods*, we ſhall ſubjoyn one or two *Corollaries*, by means whereof innumerable *Maxima* and *Minima* of their Kind may be determin'd, for the ſake of which alone theſe few things may perhaps deſerve our pains and attention.

Fig. 148. Let A B Z be a Semicircle, whoſe Centre is C, and in the Segment A D B thereof let the *Paraboloid* A F B be adapted, whoſe Exponent is $\frac{n}{m}$; alſo let A D

$= \frac{m - 2n}{m - n}$ C A, and call the *Parameter*

of the *Paraboloid* (*viz.* ſuch a right Line, ſome Power whereof drawn into ſome Power of the Abſciſs A D, ſhall be equal to ſome Power of the correſpondent Ordinate)

Plate VI.
page. 215.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dinate) *p*; then will *p* be the *Maximum* of its kind.

For draw GE any how parallel to DB, and at GE conceive a *Paraboloid* to be ſet of the ſame Nature with AFB, whoſe *Parameter* call *q*. Then becauſe the *Paraboloid* AFB touches the Circle outwardly, it will be $GF \sqsubset GE$, and ſo $GF^{m} \sqsubset GE^{m}$. That is, $p^{\overline{m-n}} \times AG^{n} \sqsubset q^{\overline{m-n}} \times AG^{n}$: Wherefore $p \sqsubset q$.

You are to obſerve that $p^{\overline{2m-2n}} = ZD^{m} \times AD^{\overline{m-2n}}$. And $q^{\overline{2m-2n}} = ZG^{m} \times AG^{\overline{m-2n}}$. Therefore $ZD^{m} \times AD^{\overline{m-2n}} \sqsubset ZG^{m} \times AG^{\overline{m-2n}}$. Wherefore $ZD^{m} \times AD^{\overline{m-2n}}$ is a *Maximum*.

EXAMPLE I.

1. Let $n = 1$, and $m = 3$. Then will $p^4 = ZD^3 \times AD = ZDq \times BDq$. Or $p^2 = ZD \times BD$. Alſo $AD = \frac{1}{2} CA$.
2. Let $n = 3$, and $m = 10$. Then will $p^{14} = ZD^{10} \times AD^4$. Or $p^7 = ZD^5 \times AD^2 = ZD^3 \times BD^4$. And $AD = \frac{4}{7} CA$.

Fig. 149. 26. Alſo let there be an equilateral *Hyperbola* whoſe Centre is C, and Axis ZA, and let a *paraboliform Figure* AFB be inſcribed in it, whoſe Exponent is $\frac{n}{m}$, and *Parameter* p, and let $AD = \frac{2n - m}{m - n} CA$. Then ſhall p be the *Maximum* of its kind.

For draw EG any how parallel to BD, and conceive a *Paraboloid*, of the ſame kind with AFB, to ſtand on EG, whoſe *Parameter* is q. Then becauſe the *Paraboloid* AFB touches the *Hyperbola* inwardly, it will be $GF^m \sqsupset GE^n$. That is, $p^{\overline{m-n}} \times AG^n \sqsupset q^{\overline{m-n}} \times AG^n$. Therefore $p \sqsupset q$.

Obſerve, that it is $p^{\overline{2m-2n}} = \frac{ZD^{m}}{AD^{2n-m}}$.

And $q^{\overline{2m-2n}} = \frac{ZG^{m}}{AG^{2n-m}}$. Therefore it ſhall be $\frac{ZD^{m}}{AD^{2n-m}} \neg \frac{ZG^{m}}{AG^{2n-m}}$. Conſequently $\frac{ZD^{m}}{AD^{2n-m}}$ is a *Minimum*.

EXAMPLE.

1. Let $n=2$, and $m=3$. Then will $p^{2} = \frac{ZD^{3}}{AD} = \frac{BD^{6}}{AD^{4}}$. And $p = \frac{BD^{3}}{ADq} = \frac{ZDq}{BD}$. Alſo $AD = CA$.

2. Let $n=3$, and $m=4$. Then will $p^{2} = \frac{ZD^{4}}{AD^{2}}$. Or $p = \frac{ZD^{2}}{AD} = \frac{BD^{4}}{AD^{3}} = \frac{ZD^{3}}{BD^{2}}$. Alſo $AD = 2\,AC$.

3. Let $n = 5$, and $m = 8$. Then ſhall $p^6 = \frac{ZD^8}{AD^2}$. Or $p^3 = \frac{ZD^4}{AD} = \frac{BD^8}{AD^5} = \frac{ZD^5}{BD^2}$. Alſo $AD = \frac{2}{3}$ CA.

ig. 150. As we have touched upon the Buſineſs of *Cyclometry* in this *Appendix*, why may we not annex by the bye a few *Theorems*, at Hand relating thereto? But firſt we muſt lay down the following general One.

If there be any Curve AGB, whoſe Axis is AD, and the right Lines BD, GE Ordinates to the ſame. The Curve AB, will have a greater Ratio to the Curve AG, than the right Line BD to the right Line GE.

For draw the right Line GH parallel to AD, and conceive the right Line BH to be divided in the Points Y, into an indefinite Number of ſmall Parts, and the right Line GE into an indefinite Number of ſmall Parts in the Points Z; and thro' the Points Y, Z draw the right Lines YM, YN, ZO, ZP parallel to AD, cutting the Curve in the Points M, N, O, P, thro' which draw the right Lines MR, NS, OT, PV parallel to BD. Then the Angle BMY (as is evident from what has been already ſhewn) is leſs than the Angle NGS, and ſo MB : BY ⊏ GN : NS. And by the like Reaſon it is NM : MR ⊏ GN : NS*. Wherefore conjoyntly it is BM + MN + NG : BY + MR + NS ⊏ GN : NS. That is, the Arch GB : BH ⊏ GN : NS. Again (by the like Reaſoning) the Ratio of GN to NS is greater than each of the Ratio's of OG to GZ, OP to OT, and AP to PV; and ſo conjoyntly it is GN : NS ⊏ Arch AG : GE. Wherefore it ſhall be GB : BH ⊏ AG : GE. And by Permutation GB : AG ⊏ BH : GE. Wherefore (*Componendo*) it is AB : AG ⊏ BD : GE.

* *Vid. App. Lect.* 12.

Fig. 151.

28. Let AMB be a Circle, whoſe Radius is CA, to which the right Line DBE is Perpendicular. Alſo let the Curve ANE be ſuch, that drawing the right Line PMN parallel to DE, (cutting the Circle in M and the ſaid Curve in N,) the right Line PN may be equal to the *Arch* AM. Alſo with the Axis AD, and Baſe DE, let a *Parabola* AOE be deſcribed: This will fall quite without the Curve ANE.

For let the right Line PN cut the *Parabola* in the Point O, and joyn the Subtenſes AB, AM. Then is DE : PN :: Arch AB : Arch AM ⊏ AB : AM :: DE :

: PO. Wherefore $PN \sqsupset PO$. Whence the Propoſition is evident.

Fig. 152. 29. Hence (from the well known *Menſuration* of the Spaces ADB, ADE) we eaſily gather this *Rule* $\frac{3CA \times DB}{2CA + CD} \sqsupset$ Arch AB.

Alſo if you put the Arch $AB = 30$ Degrees, and $2AC = 113$. By Computation according to this Rule the whole *Circumference* will come out greater than 355 leſs a Fraction of Unity.

30. Hence alſo the *Arch* AB being given, and calling AB, p, and CA, r, and DB, e; the right Sine DB may be found by this Equation $\frac{3rrpp}{9rr+pp} = \frac{12rrp}{9rr+pp}e - ee$. Or putting $k = \frac{3rrp}{9rr+pp}$; it ſhall be $kp = 4ke - ee$. Or $2k - \sqrt{4kk - kp} = e$.

Fig. 153. 31. Let AMB be a Circle, whoſe Radius is CA, and let the right Line DBE be perpendicular to it. Alſo let the Curve ANE be a Part of the Curve of the *Cycloid* belonging to the Circle AMB. And laſtly, let a *Parabola* AOE be deſcribed with AD as an Axis, and DE as a Baſe. This will fall quite within the *Cycloid*.

For draw the right Line PMON parallel to DE, cutting the Lines as you ſee, and joyn the Chords AB, AM. Then is DE : PO :: AB : AM :: Curve AE : AN $\sqsubset$ DE : PN; and ſo PO $\sqsupset$ PN. Wherefore the Propoſition is manifeſt.

32. From hence we get this Rule (from the known Dimenſions of circular and *Cycloidal* Segments) *viz.* Fig. 154.

$$\frac{2\,CA \times DB + CD \times DB}{CA + 2\,CD} \sqsubset \text{Arch}$$

AB.

Beſides, if the Arch AB be = 30 Degrees, and you put 2CA = 113. By this Rule we infer that the whole *Circumference* is leſs than 355, more by a Fraction. Fig. 152.

Hence you ſee *Metius*'s Proportion of the Diameter to the Circumference immediately ariſes from theſe two Rules.

33. Becauſe the *Cycloid* appears here out of due Order I ſhall take Notice of a Property thereof, which as far as I can perceive has eſcaped the Notice of thoſe who have largely treated of this Curve, and it is this. If the Rectangle ADEG be *compleated*, the *Space* AEG is equal to the *Segment* ADB of the Circle. I omit the Demonſtration, having deviated too far from our purpoſe already. But it may be deduced from this general *Theorem.* Fig. 153.

Let

Fig. 223. Let A M B be any Curve, whoſe Axis is A D, and Baſe B D, alſo let the Curve ANE be ſuch, that drawing P M N at pleaſure parallel to D B E and ſuppoſing the right Line T M to touch the Curve A N E let T N be parallel to the Chord A M: Then if the Rectangle A D E G be compleated, the trilineal Space A E G will be equal to the Segment A D B.

Moreover if the Segment A D B together with the Space A E G revolves about the Axis A G; the Solid generated by the Segment A D B is twice the Solid generated by A E G.

Fig. 154. 34. Let two Circles A I M G, A K N H touch one another at A, and ſuppoſe the right Line D N M be drawn any how perpendicular to the common Diameter A H G. The Segment A I M D, to the Segment A K N D will have a leſs Ratio than the right Line D M has to the right Line D N.

For let A R be perpendicular to A G, and equal to A H; and joyn H R, meeting the right Line M D in X; draw the right Line G X S. Then conceive an Ellipſis A L N G to be deſcribed thro' N to the Axis A G with the *Parameter* A S; this (as is enough evident) will fall quite within the Arch A K N. But the Segm. A I M D: Segm. A L N D :: D M :D N. Therefore the Segm. A I M D : Segm. A K N D ⊐ D M : D N.

35. Let DEC, DFC be Segments of Fig. 155. Circles, whoſe common Chord is DC and Axis GFE. I ſay the Segment DEC has a leſs Ratio to the Segment DFC, than the Axis GE has to the Axis GF.

For draw the right Light NML any how parallel to GE, and let fall the perpendiculars MX, LY, to GE. Then it may be eaſily demonſtrated that YE ⊏ XF; and ſo YE + YG : XF + XG ⊏ YG : XG. That is, GE : GF ⊏ NL : NM. And the ſame Reaſon holds with regard to other Parallels to GE. Therefore GE : GF ⊏ Segm. DEC : Segm. DFC.

From theſe *Theorems* may be gotten Rules alike to thoſe above laid down. But I ſeem to hear you crying out ἄλλην δρῦν βαλάνιζε, *Treat of ſomething elſe.*

LECTURE XII.

WE are proſecuting the Affair in Hand, which that we may contract and avoid Prolixity, as much as may be, obſerve this *common Preparation*, *viz.* the Line A B is always ſome one of the *Curves* treated of hereafter, whoſe *Axis* is AD; all the right Lines BD, CA, MF, NG being perpendicular Ordinates to it; ME, NS, C B parallel to it; the Point M taken at pleaſure; the Line MN infinitely ſmall; the right Line *ab* equal to the Curve A B, *am* equal to the Curve A M, *mn* equal to the Arch MN, and the Ordinates to the right Line *ab* perpendicular to the ſame, this being granted,

1. Let MP be perpendicular to the Curve AB; and let the Lines KZL, *a p d* be ſuch, that FZ be equal to MP, and *mp* be equal to MF: Then will the Space *abd* be equal to the Space ADLK. Fig. 156. 157.

For the Triangles MRN, PFM, are ſimilar, and ſo MN : NR :: PM : MF. Wherefore MN × MF = NR × PM, that is (ſubſtituting equals) *mn* × *mp* = FG × FZ, or the Rectangle *mt* = Rectangle FH. But the Space *abd* does not differ from an indefinite Number of Rectangles, ſuch as *ms*, and the Space ADLK is equivalent to the ſame Number of Rectangles, ſuch as FH. Therefore the thing propoſed is evident.

2. Hence, if the Curve AMB revolves about the Axis AD; the *Superficies* generated will be to the *Space* ADLK, as the *Circumference* of a Circle is to the *Radius*. Therefore if the Space ADLK be known; the ſaid Superficies will be known too. The Reaſon aſſign'd for this has been given ſome time ago. Fig. 156.

3. From hence may be had the *Menſuration* of the *Superficies* of a *Sphere*, that of each of the *Spheroids*, with that of *Conoids* too. For if AD be the Axis of a *conick* Section generating theſe Figures; the Line KZL will be always ſome one of the *conick* Sections, of no great difficul-

ty to determine. I do but juſt hint this, becauſe this Buſineſs is now well known and publiſhed.

Fig. 156. 157 4. The ſame remaining, let the Curve A Y I be ſuch that the Ordinate F Y be a mean Proportional between the anſwerable ones F M, F Z; the *Solid* generated by the *Rotation* of the Space *a d b* about the Axis *a b* will be equal to the *Solid* generated by the Space A D I revolving about the Axis A D.

For it is $MN : NR :: PM : MF :: PM \times MF : MFq :: FZ \times FM : MFq$. Whence $MN \times MFq = NR \times FZ \times FM$. That is, $mn \times mpq = NR \times FYq$. Whence the thing propoſed is evident.

5. In like manner we gather, if F Y be a *Bimedial* between F M, F Z; the Sum of the *Cubes* of the Ordinates, as *mp* (of the Curve *a p d*,) to the right Line *a b*, is equal to the Sum of the *Cubes* of the Ordinates (of the Curve A Y I) to the right Line A D. This is alſo true of other *Powers*.

Fig. 156. 6. Moreover, things remaining as before, Let the Curve V X O be ſuch, that E X be equal to M P, and the Curve *o x y* ſuch, that *m x* be equal to P F; the Space *a o y b* will be equal to the Space D V O B.

For it is $MN : MR :: MP : PF$: and ſo $MN \times PF = MR \times P$. That is, $mn \times mx = ES \times EX$ or the Rectang. ET = Rectang. ms. Wherefore the thing propounded is evident.

Fig. 156.

7. Hence we may obſerve, as a *Corollary* to this. That if the Curve AB be a *Parabola*, whoſe Axis is AD, and *Parameter* R: the Curve VXO will be an *Hyperbola*, whoſe *Centre* is D, and *Axis* DV, and whoſe *Parameter* is equal to the Axis R. Since $EXq = (PMq = PFq + FMq = \frac{Rq}{4} + FMq = \frac{Rq}{4} + DEq =)\ DVq + DEq$. Alſo the Space $abyo$ will be a *Rectangle*. Becauſe every Ordinate mx is equal to $\frac{R}{2}$. Therefore it is manifeſt that if the *Hyperbolick Space* DVOB be given; the Curve AMB is given too; and contrariwiſe. But this by the bye.

Fig. 157.

8. We may obſerve alſo that all the Squares together of the Ordinates (of the Curve oxy) to the right Line ab, are equal to all the Rectangles under PF, EX apply'd to the right Line DB; all the Cubes of mx equal to $PFq \times EX$; and ſo on.

Fig. 157. 9. We may further take Notice, that continuing out PMQ, and making FZ equal to PQ, and $mp = AQ$: the Space *abd* is equal to the Space ADLK.

For becauſe $MN : NR :: PM : MF :: PQ : QA$; it ſhall be $MN \times QA = NR \times QA$. That is, the Rectang. *mt* = the Rectang. FH.

Fig. 158. 159. 10 Again, let the right Line MT touch the Curve AB, and let the Curves DXO, *apd* be ſuch, that EX be equal to MT, and *mp* equal to MF; then ſhall the Space *abd* be equal to the Space DXOB.

For $MN : MR :: MT : MF$. Wherefore $MN \times MF = MR \times MT$. That is $mn \times mp = ES \times EX$

Fig. 158. 11. Hence again, the Superficies of a *Solid* generated by the Rotation of the *Space* ABD about the Axis AD is to the *Space* DXOB, as the Circumference of a Circle is to the Radius. Therefore if this be known, that will be ſo too: And ſo we may again get the Dimenſions of the Superficies of *Conoids* and *Spheroids*.

Fig. 158. 159. 12. If the Line DYI be ſuch, that $EY = \sqrt{EX \times MF}$: the *Solid* produced by the Rotation of the *Space* *abd* about the Axis *ab*, is equal to the *Solid* produced by the Rotation of the Space DBI about the Axis DB.

For it is $MN : MR :: MT \times MF : MFq :: EX \times MF : MFq :: EYq : MFq$. Wherefore $MN \times MFq = MR \times EYq$. That is, $mn \times mpq = ES \times EYq$.

13. After the like manner may the *Sums* of the *Cubes* or other *Powers* of the *Ordinates* mp, be compared with the Spaces apply'd to the right Line DB.

14. Let moreover the Lines AZK, Fig. 158. *axy* be ſuch, that FZ be equal to MT, 159. and mx equal to TF; then will the *Space* *aby* be equal to the *Space* ADK.

For $MN : NR :: MT : TF$. That is, $mn : FG :: FZ : mx$. Wherefore $mn \times mx = FG \times FZ$.

15. Alſo the *Sum* of the Squares of the Fig. 158. Ordinates mx is equal to the *Sum* of the 159. Rectangles under TF, FZ. And the *Sum* of the Cubes of mx is equal to $TFq \times FZ$. Suppoſing the computation to be made at the right Line AD. And the like is to be underſtood of other Powers.

16. Suppoſe again, the right Line QMP Fig. 160. to be perpendicular to the Curve AMB, 161. and let the right Line *bd* be equal to BD, and compleat the Rectangle *abdz*. Then let the Curve KZL be ſuch, that FZ be equal to QP. Then will the Rectang. *abdz* be equal to the Space ADLK.

For it is MN : NR :: (PM : MF ::) PQ : IF. Therefore MN × IF = NR × PQ. That is, $mn \times mx =$ FG × FZ. Therefore the thing is manifeſt. Hence if the Space AKLD be known; the Quantity of the Curve AMB will be known too.

Fig 160. 161. 17. Alſo ſuppoſing the right Line MTY to touch the Curve AMB, making $bc =$ BC, and compleating the Rectangle $abcy$, let the Curve OXX be ſuch, that FX be equal to TY: Then ſhall the Space ADOXX infinitely continued, be equal to the Rectangle $abcy$.

For MN : NR :: YT : DA. That is, mn : FG :: FX : mt. And $mn \times mt$ = FG × FX. Wherefore the thing is manifeſt.

Hence again, if the Space ADOXX be known, the Curve AMB will be known too.

Fig. 160. 161. 18. So likewiſe, if you take any determinate Line R, and making the right Line $bd =$ R, if the Curve OXX be ſuch, that MF . MP :: R : FX: the *Rectangle* $abdz$ will be equal to the Space ADOXX, and thence we get this Space, tho' the Curve be altogether unknown.

For MN : NR :: MP : MF :: FX : R; and ſo MR × R = NR × FX, or $mn \times mx$ = FG × FX.

We may add many more ſuch like Propoſitions, but I am afraid theſe here will ſeem more than enough.

19. Yet we may obſerve, at leaſt, that all theſe Propoſitions are equally true, and demonſtrated after the ſame manner, when the convex Side of the Curve A M B is next to the right Line A D.

20. From what has been ſhewn eaſily flows a Method of laying down *theorematical* Curves, which admit of ſome kind of Menſuration, and this you do thus,

Take any right angled right lin'd *Trapezium* having its Area ſome how or other known, contained under the two Parallels A K, D L, the right Line A D, and any right Line K L, and let another A D E C be ſo referred to this, that drawing any right Line F H parallel to D Z, cutting the Lines A D, C E, K L in the Points F, G, H, and aſſuming ſome determinate right Line Z, let the Square of F H be equal to the Squares of F G and Z. Alſo let the Curve A I B be ſuch, that continuing out the right Line G F I to it, the Rectangle under Z and F I be equal to the Space A F G C. Then will the Rectangle under Z and the Curve A B be equal to the Space A D L K. Fig. 162. Fig. 163.

The Method is juſt the ſame, if the right Line AK be even infinite.

Fig. 162. *Exam.* 1. Let KL be a *right Line*; then the Curve CGE ſhall be an *Hyperbola.*

Fig 163. 2. Let the Line KL be an *Arch* of a *Circle*, whoſe Centre is D, and let AK be $= Z$: then ſhall the Curve AGF be a *Circle*, and the Curve $AB = \frac{AD}{2} + \frac{DL}{2AK}$ Arch KL.

Fg. 164. 3. Let the Line KL be an *equilateral Hyperbola* whoſe Centre is A, and Axis $AK = Z$. Then will CGE be a right Line, and the Curve AB a *Parabola.*

Fig. 163. 4. Let the Line KL be a *Parabola*, whoſe Axis is AD: the Curve CGE will be a *Parabola* too; and the Curve AB ſome one of the *parabolick* Kind.

Fig. 165. 5. Let the Curve KL be one of the *parabolick Kind*, *viz.* ſuch that $FH = \sqrt{\frac{Z^3}{AF}}$.

Then will the Curve AB be a *Cycloid*, belonging to a Circle, whoſe Diameter is equal to Z.

With due Circumſpection you your ſelves will perhaps find out more elegant Examples.

APPENDIX I.

I Come now at laſt to touch upon Particulars by Way of *Appendix*, of which it would be no difficult Matter to compile a large Volume ariſing from theſe our general Propoſitions, tho' indeed I had no Inclination to do ſo, had not a Friend of mine deſired me, and approved of what is here ſubjoyn'd, relating to the *Tangents* and *Secants to a Circle*, as Conſectaries for the moſt Part flowing from what has been already ſaid.

A common Preparation.

Let ACB be a *Quadrant* of a Circle, Fig. 166. and let the right Lines AH, BG touch

it,

Fig. 167. it, and in HA, AC continued, take AK, CE each equal to the Radius CA; deſcribe an *Hyperbola* KZZ thro' K with the *Aſymtotes* AC, CZ, and another *Hyperbola* LOE thro' E with the *Aſymtotes* BC, BG. Alſo take the Point M in the Arch AB at pleaſure, thro' which draw the right Line CMS, and the right Line MT touching the Circle, and let CMS meet the Tangent AH in S. Alſo draw the right Line MFZ parallel to BC, and the right Line MPL parallel to AC. Moreover let the right Line ab be equal to the *Arch* AB, and am equal to the Arch AM, and let the right Lines ac, $xmps$ be perpendicular to ag, whereof $ac = AC$, $mx = AS$, $ms = CS$, and $mp = MP$.

I. The right Line CS is equal to the right Line FZ; and ſo the Sum of the *Secants* belonging to the *Arch* AM, and apply'd to the right Line AC is equal to the *Hyperbolick Space* AFZK.

For it is $CF : CA :: (CM : CS ::) CA : CS$. And ſo $CF \times CS = CAq$. Alſo $CF \times FZ = CA \times AK = CAq$. Therefore $CS = FZ$.

Fig. 167. II. The Space amx, that is, the Sum of the Tangents in the Arch AM apply'd to am, is equal to the *Hyperbolick Space* AFZK.

This appears from the *9th Lecture*,

3. Let the Curve AXX be ſuch that Fig. 166.
PX be equal to the Secant CS or CT. Then the Space ACPX, that is, the Sum of the Secants belonging to the Arch AM, and apply'd to CB, is equal to *twice the Sector* ACM.

For *(a)* the *Space* AFMX is twice the *(a)* 10. Lect. XI.
Segment AFM, and the Rectangle FCPM twice the Triangle FCM. Therefore the whole Space ACPX is twice the Sector ACM.

This is very evident too, from the 16*th* Number of the 12*th Lecture.*

4. Let the Curve CVV be ſuch, that Fig. 166.
PV be equal to the Tangent AS. Then will the Space CVP, that is, the Sum of the Tangents to the Arch AM, and apply'd to the right Line CB, is equal to one half the *Square* of the *Subtenſe* AM.

This evidently follows from Num. 7. of the 11*th Lecture.*

5. If you take CQ = CP, and draw QO parallel to CE, meeting the *Hyperbola* LE in the Point O; the *Hyperbolick Space* PLOQ drawn into the Radius CB, or the *Cylindrical* one having the Baſe Fig. 166.
PLOQ, and Altitude BC, is twice the *Sum of the Squares* of the right Lines CS or PX belonging to the Arch AM, and apply'd to the right Line CB.

For becauſe $PL : QO ::$ ($BQ : BP$; that is, $::$) $BC + CP : BC - CP$; it will be *(Componendo)* $PL + QO : QO :: 2BC : BC - CP$; it is alſo $QO : BC :: BC : BC + CP$. Therefore (joyning the equal Ratio's) it is $PL + QO \times QO : QO \times BC :: 2BC \times BC : BC - CP \times BC + CP$; that is, $PL + QO : BC :: 2BCq : BCq - CPq$. That is, $:: 2BCq : PMq$. But it is $PXq : BCq :: BCq : PMq$ or (doubling the antecedents) $2PXq : BCq :: 2BCq : PMq$. Therefore $PL + QO : BC :: 2PXq : BCq$. Or $PL \times BC + QO \times BC : BCq :: 2PXq : BCq$. And ſo $PL \times BC + QO \times BC = 2PXq$. Conſequently BC drawn into every $PL + QO$ is equal to every anſwerable Square PXq. Therefore the Propoſition is manifeſt.

Fig. 167. 6. Hence the Space $acsm$, that is, the Sum of the Secants in the Arch AM apply'd to ab is equal to the one half of the *Hyperbolick Space* $PLOQ$.

For take the indefinite ſmall Arch MN, and the right Line mn equal to it, and draw the right Line NR parallel to AC. Then is $MN : MR ::$ ($MC : CF :: CS : CA :: PX : CA$) $:: PXq : PX \times CA$. And ſo $MN \times PX \times CA = MR \times PXq$. Or $mn \times ms \times CA = MR \times PXq$. But (from the laſt Number) the Sum of all $MR \times PXq$ is the half of the Space P L O Q drawn into C A. Therefore all the Rectangles $m\ n + m\ s$ drawn into C A will be equal to the ſame. Wherefore $a\,c\,s\,m$ (equal to all the Rectangles $m\,n \times m\,s$) is equal to the one half of the Space P L O Q.

7. All the Squares of the right Lines $m\,s$ apply'd to the right Line $a\,m$ are equal to C A × C P × P X, that is, the *Parallelepipedon* with the right angled Baſe A C P D and Altitude C S. Fig. 167.

The Demonſtration of this Propoſition, altho' eaſy, I ſhall paſs over, as requiring another Scheme and a larger Proceſs than moſt of the reſt; neither does it ſeem to be a Thing of much Moment.

8. Let the Curve A Y Y be ſuch, that F Y be equal to A S. Then if the right Line Y I be drawn parallel to AC. The Space A C I Y Y A, that is, the Sum of the Tangents belonging to the Arch A M, and apply'd to the right Line A C, together with the Rectangle F C I Y is equal to one half of the *Hyperbolick Space* P L O Q. Fig. 166.

For the *Space* $a\,c\,p\,m$ *(a)* is equal to the *Rectangle* A C P D, that is, to the *Rectangle* F C I Y. For it is C A : A S :: C F : F M. Or C A : F Y :: C F : C P. And ſo C A × C P = F Y × C F. Alſo the Space $c\,p\,s$, that is, all the right Lines T F apply'd to $a\,b$, every one that do belong

(a) 1. Lect. XII

(b) 14. Lect. XII. long to the Arch AM, is *(b)* equal to the *Space* AFY; and ſo the *Space* ACIYA is equal to the *Space* *acsm*. That is, (as but now demonſtrated) equal to the *half* of the *Hyperbolick Space* PLOQ.

We may meaſure this otherwiſe by means of the following *Lemma*.

Fig. 168. Let there be an *equilateral Hyperbola* ERK, *viz.* having equal Axes, and let KI, KD be Ordinates to theſe Axes (CED, CI). Alſo let the Curve EVY be ſuch, that aſſuming the Point R at pleaſure in the *Hyperbola*, and drawing the right Line RVS parallel to DC, let SR, CE, SV be continual Proportionals. Then joyning the right Line CK; the Space CEIY will be the double of the *Hyperbolick* Sector KCE.

For draw RT touching the *Hyperbola*, and RH parallel to CI. Then it is CH : CE :: CE : CT. Therefore CT = SV, or HT = RV. Conſequently the *Space* EDKY is twice the *Segment* EDK. 10. Lect. XI. Alſo the Rectangle IKDC is twice the Triangle CDK. Conſequently the *remaining Space* CEYI is twice the *remaining Sector* ECK.

Fig. 169. 10. Now reſuming the circular Quadrant ACB, let CE = CA, and with the Axis AE, as *alſo Parameter* AE, let an *Hyperbola* EKK be deſcribed, and ſuppoſing the Curve AYY to be ſuch, that drawing any Ordinate MFY; FY be equal to the Tangent AS, draw the right Line YIK cutting CZ in the Point I, and the *Hyperbola* in K, and joyn CK. Then the *Space* ACIYA is twice the *Hyperbolick Sector* ECK.

For it is $CIq : CAq :: ASq : CAq$ Fig. 169. $:: FMq : CFq :: CAq - CFq : CFq$. And *(Componendo)* $CIq + CAq : CAq :: CAq : CFq$. That is, (from the Nature of the *Hyperbola)* $IKq : CAq :: CAq : CFq$, or $IK : CE :: CE : IY$. And therefore it is evident from the precedent *Lemma*, that the Space ACIYA is twice the Sector ECK.

11. *Coroll.* Hence if a *Conchoid* AVV be deſcribed with the Pole E, *Chord* CB, and *Sagitta* CA, and YFM continued out to V meets it; we ſhall have $MV = FY$; and ſo the *Space* AMV is twice the Space AFY.

12. Therefore the Menſuration of ſuch *Conchoidal Spaces* as theſe are known.

I do not know whether we may add this *Corollary* to *the Work.*

13. Let the right Line AE be perpendicular to the right Line RS, and $CE =$ Fig. 170. CA, and let there be two *Conchoids* AZZ, EYY, (inverſely ſituated with regard to each other) deſcribed from the ſame *Pole* E, with the *common Ruler* RS, and from E draw

E draw any how the right EYZ (interſecting the Lines as you ſee.) Alſo let EKK be an *equilateral Hyperbola*, whoſe Centre is C, and *Semi-Axis* CE; draw IK parallel to AE, and joyn CK. Then will the *Quadrilateral Space* AEOYZPA (comprehended by the right Lines AE, YZ and the *Conchoids* EOY, APZ) be equal to four times the *Hyperbolick Sector* ECK.

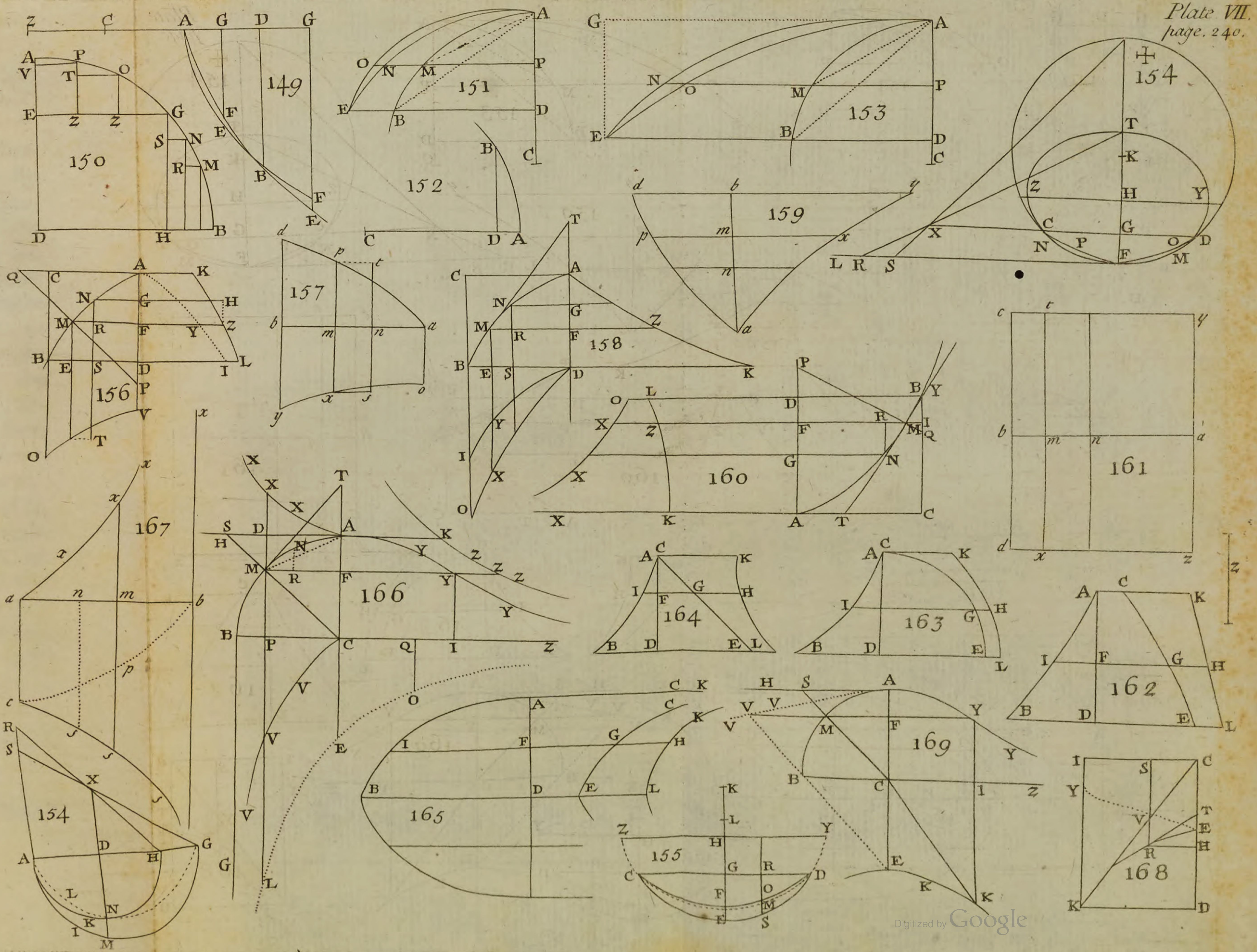
For if with the *Centre* E, you deſcribe thro' C an *Arch of a Circle* CX; it is eaſy to gather from what has been ſaid that the *Space* APZIC is equal to twice the *Hyperbolick Sector* ECK, together with the *circular Sector* CEX. Alſo the *Space* EOYIC is equal to *twice* the *Sector* ECK, leſs the Sector CEX.

This alſo we may eaſily infer: draw ZF, YG parallel to CS, and continue out GYL, LIH. Then becauſe $IY = IZ$, it is $FZ + GY = 2\,CI$. And the *Trapezium* FGYZ = *Rectang.* EGLH $= 2CG \times CI$. Therefore the *Corollary* is manifeſt.

You may obſerve by the way, if you will, that drawing AT parallel to CS, and continuing out EZT, if you put N = 2 *Triang.* CEI — 2 *Sect.* ECK; then will the *Space* EZT + EOYE be = 2 N.

For

Plate VII.
page. 240.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For $N + CXI =$ *Space* AZT, and $N - CXI =$ *Space* $EOYE$.

14. We will likewiſe ſubjoyn the following like Menſuration of the *Ciſſoidal Space.*

Let AMB be a *Semicircle*, whoſe Centre is C, and let AH be a Tangent to the ſame. Alſo let AZZ be a *Ciſſoid* agreeing to the ſame, whoſe Nature is ſuch, that taking a Point M at pleaſure in the Circumference AMB, and a right Line BMZ be drawn thro' it, as alſo the right Line MFZ cutting the Curve in Z, we have $MZ = AS$. Then if the right Line *ab*, you take *am* equal to the Arch AM, and to *am* apply the Perpendiculars *mx* equal to the *verſed Sines* AF of the *Arches* AM. The *trilineal Space* MAZ will be twice the *Space axm.* Fig. 171.

For take the indefinitely *ſmall Arch* MN, and let *mn* be equal to it. Draw the right Line NR parallel to AB, and joyn CM. Now it is $AS : AB\ (2\,CM) :: (FM : FB ::)\ AF : FM$. And $2\,CM : 2\,MN :: (CM : MN)\ FM : NR$. Therefore it ſhall be (by equality) $AS : 2\,MN :: AF : NR$; and ſo $NR \times AS = 2\,MN \times AF$. That is, $NR \times MZ = 2\,mn \times mx$. Wherefore the *Space* MAZ is equal to *twice* the *Space amx.*

Fig. 172. Therefore becauſe the Menſuration of the *Space amx* is well known; we have the Dimenſion of the *Ciſſoidal Space* MAZ. As any one who is willing may try.

I ſhall conclude this *Appendix* with the following *Conſectary.*

15. Let ACB be a *Quadrant* of a Circle, and let AH, BG touch the Circle. And let the Curves KZZ, LEO be *Hyperbola*'s, the ſame, as *(a)* above. Let any Arch AM of it be conceived to be divided, in the Points N, into parts indefinitely many in Number, thro' which draw the *Radii* CN, and let the right Lines NX meet theſe in the Points X; the *Sum* of the *right Lines* NX, drawn into the *Radii*, is equal to the *Space* $\frac{AFZK}{Rad.}$; and the *Sum of the right Lines* NX, into the Parallels to AS, is equal to the *Space* $\frac{PLQO}{2\,Rad.}$ for the Triangle XMN is ſimilar to the Triangle CAC; and ſo XM : MN :: AS : CA. And XN : MN :: CS : CA. Therefore XM $= \frac{MN \times AS}{CA}$, and XN $= \frac{MN \times CS}{CA}$. And ſo in the reſt. Therefore the Thing is manifeſt, from *n.* 2. and 7 of this Chapter.

Fig. 173.

(a) 7. and 12.

APPEN-

APPENDIX II.

COnſulting Brevity, but Perſpicuity chiefly, carried me to lay down our Propoſitions demonſtrated after a direct Method, being ſuch that the Truth, as I apprehend, is not only ſufficiently confirm'd, but likewiſe the Origine and Nature thereof made more evident. But leſt ſome who are not much accuſtomed to this Way of Reaſoning ſhould be at a ſtand, I ſhall add the few things following, that both confirm our Method, and will alſo aſſiſt them to demonſtrate our Propoſitions *ad abſurdum*, without any great Difficulty.

1. Let there be any Number of *Ratio*'s A to X, B to Y, C to Z each greater than a given *Ratio* R to S; the Ratio of all the *Antecedents* taken together to all the *Conſequents* taken together will be greater than the Ratio R to S.

$$\begin{array}{c|c} A : X. & A : M. \\ B : Y. & B : N. \\ C : Z. & C : O. \end{array}$$

For let the Ratio's A to M, B to N, C to O be each equal to the Ratio R to S. Therefore $X \sqsupset M$, and $Y \sqsupset N$, and $Z \sqsupset O$. Conſequently it is manifeſt that $A + B + C : X + Y + Z \sqsubset A + B + C : M + N + O$. That is, $A + B + C : X + Y + Z \sqsubset R : S$.

2. Hence it is manifeſt, if there be any Number of Ratio's greater than a given Ratio; the Sum of the Antecedents to the Sum of the Conſequences will have a greater Ratio than the given Ratio.

3. Let there be any Curve ADB, whoſe Axis is AD, and let BD be an Ordinate Fig. 174. to the ſame, let BT touch the Curve, and let BP be an indefinitely ſmall Part of the right Line BD, and draw the right Line PO parallel to DT, cutting the Curve in N. I ſay the Ratio of PN to NO is greater than any given Ratio, ſuppoſe R to S.

For let DE : ET : : R : S, and joyning the right Line BE let it cut the Curve in G, and the right Line PO in K; and thro' G draw FH parallel to DA. Then becauſe BP is put indefinitely ſmall, it is $BP \sqsupset BF$; and ſo $PK \sqsupset PN$ (for the Chord BG falls quite within the Curve.) Therefore $PN : NO \sqsubset PX : KO :: DE : ET :: R : S$.

Fig. 175.

4. Hence, if the Baſe DB be divided into an indefinite Number of Parts in the Points Z; and thro' theſe be drawn right Lines parallel to DA cutting the Curve in the Points E, F, G, and thro' theſe be drawn the Tangents BQ, ER, FS, GT meeting the Parallels ZF, ZE, ZG, DA in the Points Q, R, S, T: The right AD will have a greater Ratio to all the Abſciſs's EQ, FR, GS, AT taken together, than any aſſignable Ratio.

For draw the right Lines EY, FX, GV parallel to BD. Then the right Lines ZE, YF, XG, VA will have a greater Ratio to the right Lines EQ, FR, GS, AT, (each to each reſpectively in a right Line to one another) than any given Ratio. Therefore all theſe together to all thoſe, will have a Ratio greater than any aſſignable one. That is, the right Line AD has this Ratio to $EQ + FR + GS + AT$.

5. Hence by the way, the Aggregate of EQ, FR, GS, AT, are equivalent to nothing; or the right Lines ZE, ZQ; and ZF, YR, &c. are equal. Alſo the Particles of the Tangents BQ, ER, &c. And the correſpondent ſmall Portions BE, EF, &c. of the Curve are equal,

and may be conſider'd as coinciding. what evidently correſponds with theſe may be ſafely taken as ſuch.

Fig. 176. 6. Again, let there be any Curve A B, whoſe Axis is A D, and D B an Ordinate to it, and conceive D B to be divided into an indefinite Number of equal Parts in the Points Z, thro' which draw right Lines parallel to A D, cutting the Curve in the Points X, which the right Lines M E, N F, O G, P H drawn from X parallel to B D do meet. Alſo let the Figure A B D M X N X O X P X R A circumſcribing the Segment A D B, contain'd under the right Lines A D, B D and the Curve A B, be greater than ſome Space S. I ſay the Segment A D B is not leſs than S.

For if poſſible, let A D B be leſs than S by a Rectangle A D L K. Then becauſe A R is indefinitely ſmall, and ſo leſs than A K, it is evident that the Rectangle A D Z R is leſs than the Rectangle A D L K. Alſo it is evident that the Segment A D B together with the Rectangle A D Z R is greater than the circumſcribing Figure. For the Rectangle A D Z R is equal to Fig. 176. the Rectangles R H, P G, O F, N E, M Z, and ſo greater than the trilineal Figures A X R, X X P, X X O, X X N, X B M. Therefore the Segment A D B together with the Rectangle A D L K is much greater than the circumſcribing Figure. That is, the Space S greater then the circumſcribing Figure. Which is contrary to the Suppoſition.

7. In like manner if HXGXEXZDH be an inſcribed Figure, leſs than ſome Space S: I ſay the Segment ADB is not greater than S.

For if you will have it to be greater, let the Exceſs be equal to the Rectangle ADLK, which will be, like as before, greater than the Rectangle ADZR. But the Segment ADB, leſſened by the Rectangle ADZR, is leſs than the inſcribed Figure. Therefore the Segment ADB, leſſen'd by the Rectangle ADLK is much leſs than the inſcribed Figure. That is, the Space S leſs than the inſcribed Figure. Which is contrary to the Suppoſition.

8 Hence, if there be any Space, as S, the circumſcribing Figure whereof is equal to ADBMNOPRA, as alſo the inſcribed Figure, equal to the Figure HGFEZDH, it is plain that the ſaid Space S is equal to the Segment ADB.

For we have juſt now ſhewn that this, can neither be greater nor leſs than that.

What has been here ſaid may alſo be accommodated to other Ways of Circumſcription and Inſcription. Of which it may be ſufficient to have juſt hinted.

A Method to Meaſure the Superficies of Cones.

Fig. 177. Let AMB be any Curve, whoſe Axis is AD, in which let C be an aſſum'd Point, and let the right Line BD be an Ordinate to AD. From any Point M taken in the Curve draw the right Line ME touching the Curve, and from C let fall CG perpendicular to ME. Likewiſe let the determinate Line CV be perpendicular to the Plane DAB, and joyn VG, which will be perpendicular to MG: ſince if you draw CH parallel to GM, it is plain that CH is at right Angles to the Plane GVG, and ſo GM will be perpendicular to it alſo.

Now let the Line RS be ſuch, that drawing the right Line MIX parallel to AD, cutting the Ordinate BD in I, and the right Line RS in X, it be MP:ME :: VG:IX, or let the Line AL be ſuch, that drawing MPY parallel to BD, cutting the Axis AD in P, and the Line AL in Y, we have PE; ME :: VG : PY. Then will either of the Spaces BRSD, or ADL be twice the Superficies of the Cone generated by the Motion of a right Line continually paſſing thro' V, and at the ſame time moving along the Curve AMB.

For take the indefinitely ſmall Part MN of the Curve, and thro' N draw the right Line $NOKT$ parallel to AD, and the right Line NQZ parallel to BD, cutting the other Lines of the Scheme, as you ſee, and joyn the right Lines VM, VN. Then it is $MO : MN :: MP : MF :: VG : IX$. Wherefore $MN \times VG = MO \times IX = IK \times IX$. Alſo it is $NO : MN :: PE : ME :: VG : PY$. Therefore $MN \times VG = NO \times PY = QP \times PY$. But $MN \times VG$ is twice the Triangle MVN. Conſequently $IK \times IX$ and $QP \times PY$ are each twice the Triangle MVN. And this is always ſo. Therefore the Thing propoſed is evident.

An *EXAMPLE.*

Let the Curve AMB be an *equilateral* *Hyperbola*, and C the *Centre*, and let $CV = CA$ be $= v$, and $CP = x$. (For of theſe things a *Computation* moſt commonly ſhortens the Buſineſs) and joyn MC. Then it is evident that $EC = \frac{rr}{x}$, and $MCq = 2xx - rr$. For $PMq = xx - rr$. Alſo it is $MCq : CPq :: MEq : MPq$. That is, $MCq : CPq :: ECq : CGq$. Fig. 177.

That is, $2xx - rr : xx :: \frac{r^4}{xx} : CGq = \frac{r^4}{2xx - rr}$. Wherefore $VGq = \frac{r^4}{2xx - rr} + rr = \frac{2rrxx}{2xx - rr} = \frac{VAq \times CPq}{MCq}$. Or $VG = \frac{VA \times CP}{MC}$. Therefore $VG : VA :: (CP :: MC) :: MP : ME$. So that in this Caſe, I X is every where equal to V A, and the Line R S a ſtrait one. Conſequently the Rectangle BRSD, is twice the *conical Superficies* AMBV.

We have been furniſhed with this elegant Example, by the Ingenious and very Learned *Francis Jeſſope*, Eſq; an Ornament of our College, being formerly a Gentleman-Commoner thereof, with which very ingenious Invention in this Matter, I have made bold to adorn my own, as with ſome precious Stone, tho' not indeed at his Deſire, but yet, I hope, not contrary to his Inclination.

PROP. I.

If from the Point E in the *Axis* A *m* of a right *Cone* A B C *p*, an indefinite right Line E C paſſes thro' the *Superficies* of the Cone, and while the Point E is at reſt, let the right Line E C move round until it returns to the ſame Place from whence it moved, ſo that ſome Part of it always cuts the Superficies of the Cone, ſuppoſe thro' the *Hyperbola* C F D and right Lines D A, A C ſituate in the Superficies of the Cone. The Solid contain'd under the Superficies, or Surfaces generated by the Line E C thus moved, and the Portion of the Superficies of the ſame Cone terminated by a Line or the Lines C F D, D A, A C, which the right Line E C deſcribes by its Motion in the Superficies of the *Cone*, will be equal to a *Pyramid* whoſe *Altitude* is equal to the Perpendicular E *n* drawn from the Point E to the ſide of the Cone, and *Baſe* equal to the ſame *Superficies* of the Cone terminated by the Line or Lines C F D, D A, A C generated by the Motion of the Line E C. Fig. 178.

For the *Solid* E C F D A C conſiſts of an infinite Number of *Pyramids* E C *o* A, E *o o* A, *&c.* of the ſame Altitude; (E *n* being the Perpendicular) all the Baſes whereof taken together will exhauſt the Superficies of the *Cone* CFD, DA, AC.

PROP. II.

Fig. 178. Let the right *Cone* ABC*p* be cut by a Plane CFD parallel to the Axis A*m*, and from the Vertex thereof to the *Hyperbola* CFD draw the right Lines AC, AD, and upon the Triangle ACD let a *Pyramid* EACD be erected having its Vertex E in the Axis of the *Cone*, and let E*d* be perpendicular to the Plane ACD, and E*n* to the ſide of the Cone.

I ſay the Superficies of the Cone terminated by the *Hyperbola* CFD and the right Lines DA, AC is to the Baſe ACD of the *Pyramid* EACD, as the Altitude E*d* of the *Pyramid* EACD is to the Perpendicular E*n*.

For becauſe the Cones ACFD, ECFD have their Virtices A and E in a Plane parallel to the Plane of the Baſe CFD, common to both Cones, therefore they are equal. Conſequently if from a Solid compounded of the Cone ACFD, and the *Pyramid* ECAD, be taken away the Cone ECFD, the Remainder will be the the Solid ECFDAC, ſuch as is deſcribed in the firſt *Propoſition*, being equal to the *Pyramid* EACD. But becauſe *the Baſes* and *Altitudes* of equal *Pyramids* are reciprocally proportional, it ſhall be as the Altitude E *d* of the *Pyramid* EACD is to the Perpendicular E *n* ſo is the Superficies of the Cone bounded by the *hyperbolick* Line CFD and the right Lines DA, AC to the Triangle ACD.

PROP. III.

Let ABC*p* be a right Cone. Cut it by a Plane, ſuppoſe the Triangle *q r t*, which Plane let cut the Axis in the Point *q*, continued out beyond the Vertex, and in the common Section with the Superficies of the Cone, let there be an *Hyperbolick Line r* S *t*. From the Vertex A of the Cone draw the right Lines A *r*, A *t*, and from the Point *q*, let fall the Perpendicular *q* X to the Side A *p* of the Cone continu'd out, and from the Point A draw AZ perpendicular to the Plane *q r t*. Fig. 178.

I ſay the *conical Superficies* comprehended under the *Hyperbolick Line r s t*, and the right Lines *r* A, *t* A is to the *hollow Hyperbolick Figure q r s t q*, as the Perpendicular AZ is to the Perpendicular *q* X.

For while the Point q is at reſt, if the right Line $q\,r$ moves along the Lines $r\,s\,t$, t A, A r; it generates three Surfaces, the hollow *Hyperbola* $q\,r\,s\,t\,q$, and the two Triangles $q\,t$ A, q A r, which, together with the conical Superficies terminated by the Lines $r\,s\,t$, t A, A r, will contain the Solid $q\,r\,s\,t$ A r. But this Solid is equal to a *Pyramid*, whoſe Altitude is equal to the Perpendicular q X. For the infinite Number of *Pyramids* q A r V, q A V V, do exhauſt the Solid $q\,r$ S t A r. But if we have a Mind to conſider the Thing otherwiſe, this Solid may be taken as a conical Figure A r S $t\,q\,r$ having the hollow *Hyperbolick* Figure $q\,r$ S $t\,q$ for a Baſe, and the Perpendicular AZ for the Altitude. Therefore making the Baſes and Altitudes reciprocal, it is as AZ to q X, ſo is the *Superficies* r S t A r to the *hollow Hyperbolick Figure* $q\,r$ S $t\,q$.

PROP. IV.

Fig. 179. Let the right Cone A B $h\,g$ be cut below the Vertex, by a Plane HFEG, thro' the Axis, and from the Point H where the Plane cuts the Axis of the Cone, let fall H K perpendicular to any ſide of the Cone, and from the Vertex A, the perpendicular AL to the Plane HFEG.

I ſay the *conical Superficies* bounded by the Lines F E G, G A, A F is to the Plane H F E G as the Perpendicular A L, is to the Perpendicular H K.

The Demonſtration is much the ſame as that of the laſt *Problem.*

APPENDIX III.

UPON re-conſidering what foregoes, I find ſome things have eſcap'd, which perhaps it may be neceſſary to add, whoſe Demonſtration any one may of himſelf eaſily gather from the Premiſes, and in ſo doing will reap both Pleaſure and Advantage extraordinary.

PROB. I.

Fig. 180. Let there be any Curve KEG, whoſe Axis is AD, and a given Point A in the ſame. It is required to find a Curve, as LMB, of ſuch a Nature, that if the right Line PEM be any how drawn Perpendicular to the Axis AD, cutting the Curve KEG in E, and the Curve LMB in M, and AE be joyn'd, and the right Line TM touches the Curve LMB, we ſhall have TM parallel to AE.

To do this you muſt proceed thus. Thro' any Point R taken in the Axis AD draw the right Line RZ perpendicular to AD, meeting the right Line EA continued out in the Point S, and in the right Line EP take $PY = RS$; thus will the Nature of the Curve OYY be determined. Then let the Rectangle under AR and PM, be equal to the Space AYYP (or $\frac{\textit{Space}\ AYYP}{AR}$); and the Curve LMMB will have the propoſed Condition.

We may obſerve by the Way, all things elſe being the ſame, that if the Curve QXX be ſuch, that when the right Line EP cuts it in X, PX is equal to AS. I ſay if it be ſuch, the Space AXXP ſhall be equal to the Rectangle under AR and the Curve LM, or $\frac{AXXP}{AR} = LM$.

EXAMPLE I.

Let ADG be a Quadrant of a Circle, Fig. 181. and any how drawing EP perpendicular to AD, ſuppoſe the Curve AMB of ſuch a Nature, that if the right Line EPM continu'd cuts it in M, and the right Line MT touches it, we ſhall have MT parallel to DE. To do this, draw AZ parallel to D G, meeting D E continu'd out in the Point S, and let the Curve A Y Y be ſuch, that if the right Line P E continued to Y cuts it, we may have $PY = AS$. Then take $PM = \frac{\text{Space } AYP}{AD}$. And the thing required will be done.

Obſerv. If the Curve Q X X be ſuch, that $PX = DS$, or if $AQ = AD$, and Q X X be an *Hyperbola* contain'd within the Angle A D G; the Curve $AM \times AD$ will be = Space A Q X P.

EXAMPLE II.

Fig. 182. Let the Curve A E G, whoſe Axis is A D, be of ſuch a Nature, that if from any Point E taken in the ſame be drawn the right Line E P perpendicular to A D, and joyning A E we may have A P, a mean Proportional between A R and A P, acccording to the Order whoſe Exponent is $\frac{n}{m}$: to find the Curve A M B which the right Line T M parallel to A E touches.

Concerning the Curve A M we may obſerve that $n : m :: AE : \text{Arch } AM$.

If $\frac{n}{m}$ be $= \frac{1}{2}$, or A E be a ſimple mean Proportional between A R and A P; A E G will be a *Circle*, and A M B a *Cycloid*;

cloid; and therefore the Menſuration of it is had from the general Law.

Theſe are alſo performed more intelligible from the *Problem* adjoyn'd.

PROB. II.

To draw a Curve; ſuppoſe A M B, whoſe Axis is A D, being ſuch, that taking any Point M in it, and drawing M P perpendicular to A D, and putting the right Line M T to touch the Curve; T P to P M may have a given Relation. Fig. 183.

Take any right Line R, and make as T P to P M, *(viz.* the Ratio given) ſo is R to P Y, taken in the right Line P M, as an Ordinate to the Axis A D, ſo that the Curve Y Y K paſſes thro' all the Points Y thus found. Then if you make P M $=$ Space $\frac{APY}{R}$. The Nature of the Curve A M B will be thence manifeſt.

EXAMPLE I.

Let A D G be a Quadrant of a Circle, whoſe Radius is equal to the aſſign'd Quantity R; and T P is to have to P M the ſame Ratio that R has to the Arch A E. Therefore ſince it is, (by Suppoſition) R : Arch A E : : R : P Y; P Y ſhall Fig. 183.

be equal to the Arch A E: and ſo $PM = \frac{APY}{R}$.

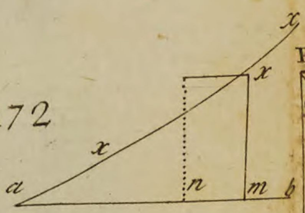
EXAMPLE II.

Fig. 184. Let A D G be a Quadrant of a Circle, and ſuppoſe T P is to have the ſame Ratio to P M as P E to R: Then P Y is equal to the Tangent of the Arch G E, and the Space A P Y Y = R × Arch A E; and ſo P M = Arch A E.

PROB. III.

Let there be any Curve Line A M B, whoſe Axis is A D and Baſe D B; to deſcribe the Curve A N E, of ſuch a Nature, that drawing the right Line M N G any how parallel to B D, cutting A N E in the Point N, the Curve A N may be equal to G M. Fig. 221.

Let the Curve A N E be ſuch, that if M T touches the Curve A M B, and N S the Curve A N E, it be $SG : GN :: TG : \sqrt{GMq - TGq}$. Then will A N E ſatisfy the Demand.



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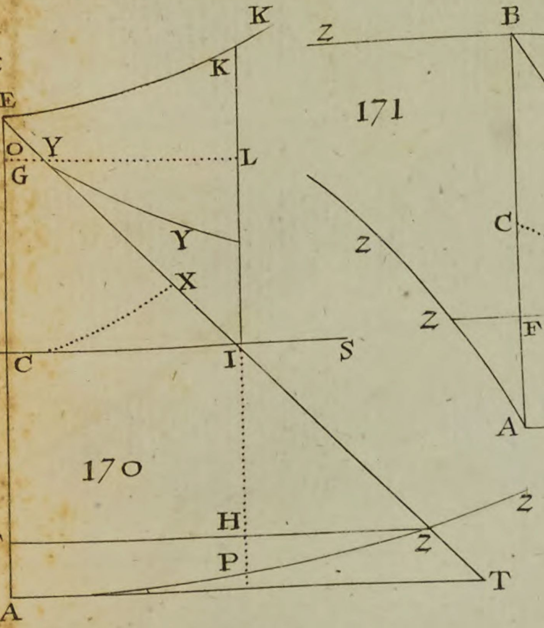

170

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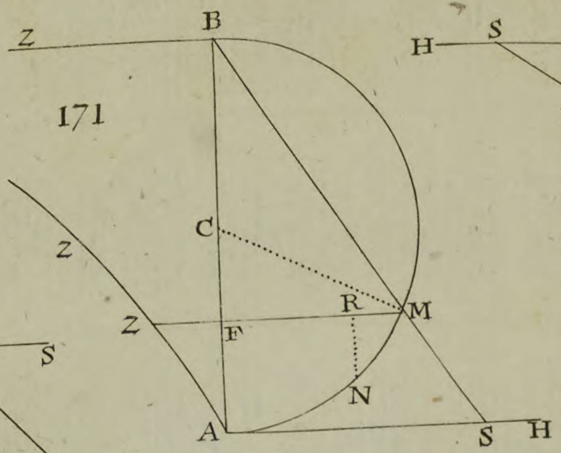

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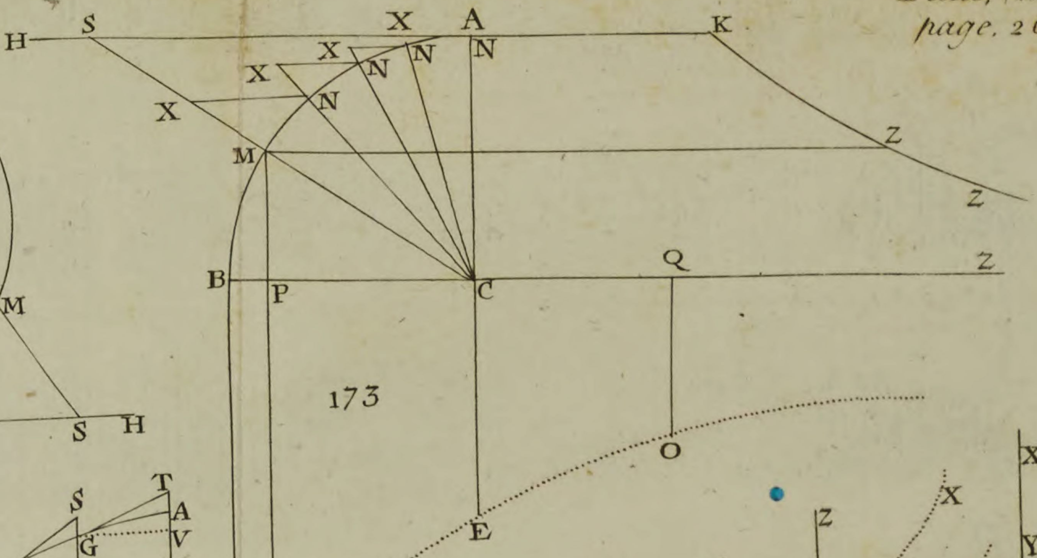

173

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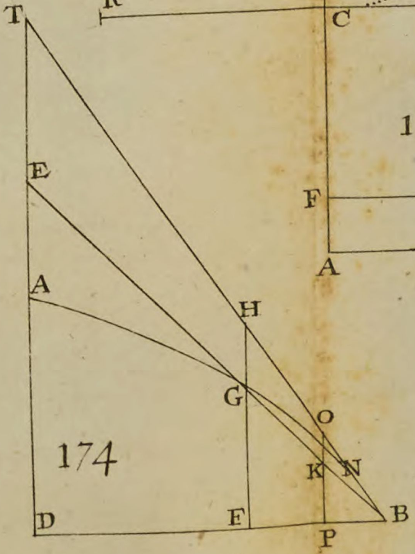

174

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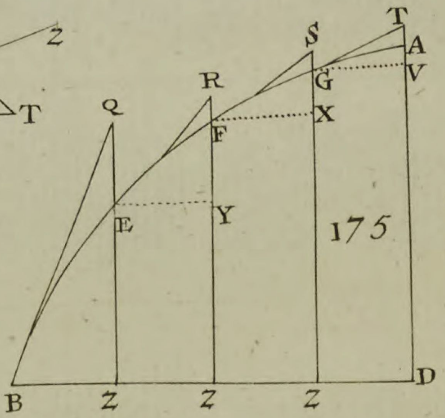

175

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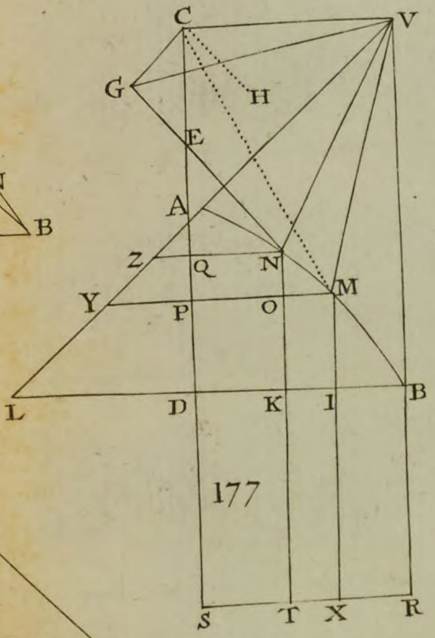

177

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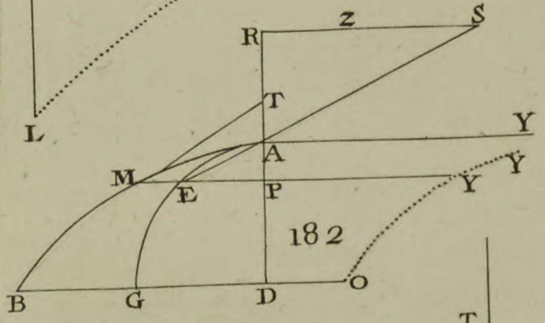

182

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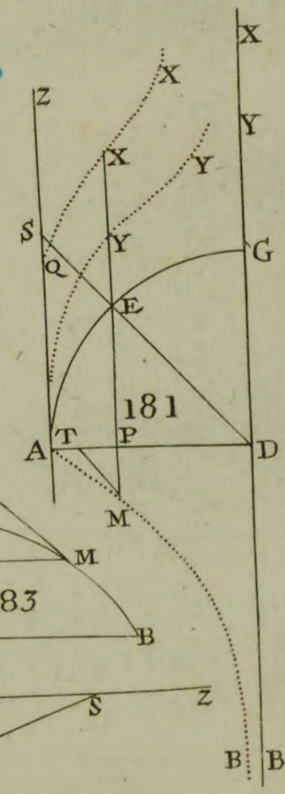

181

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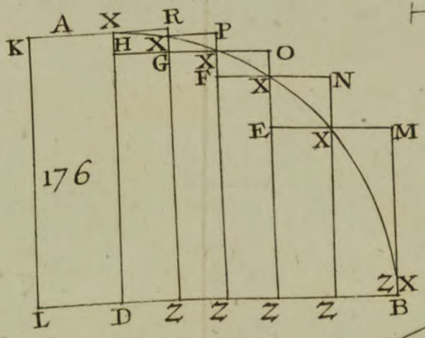

176

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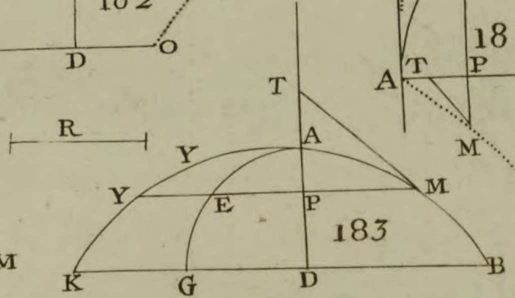

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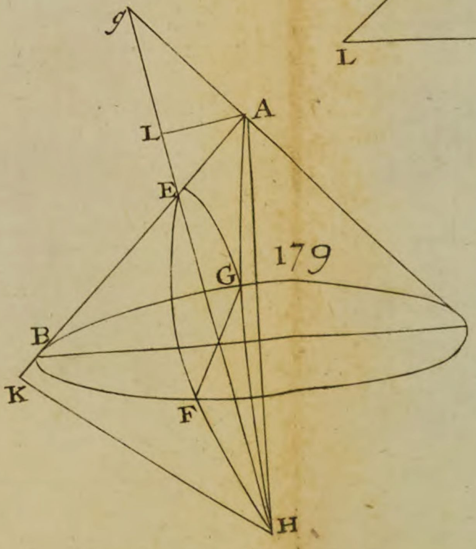

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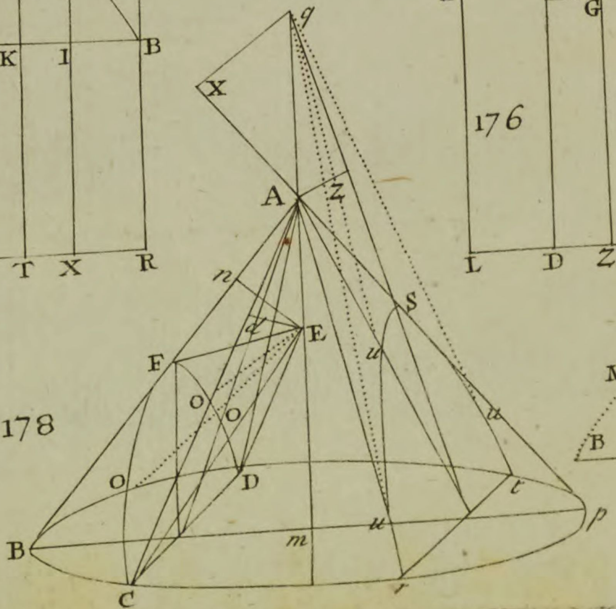

178

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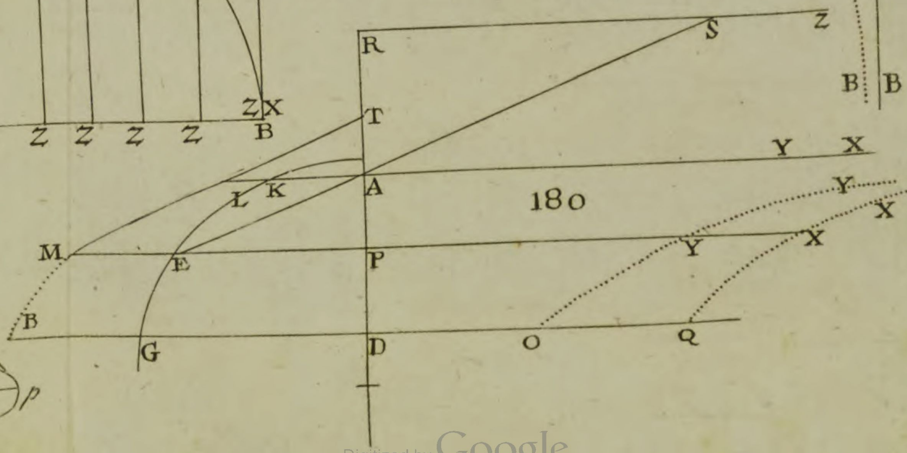

180

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PROB. IV.

The ſame Suppoſition and Conſtruction remaining; to find the Curve ANE, ſuch that the Curve AN be always equal to the intercepted right Line MN.

Let now the Curve ANE be ſuch, that it be $SG : GN :: 2 TG \times GM : GMq - TGq$. Then will ANE be the Curve wanted.

PROB. V.

Let DXX be any given Curve, whoſe Axis is DA. To find the Curve AMB of ſuch a Nature, that if the Perpendicular right Line GXM be any how drawn to AD, and SMT be a Tangent to the Curve AM; MS may be equal to GX. Fig. 222.

It is evident that the Ratio TG to TM, that is, the Ratio GD to MS or GX is given; and ſo the Ratio TG to GM is alſo given.

This ſerves to lay down Superficies whoſe Menſuration is ready at Hand. For drawing ME parallel to AD, the Superficies of the Solid generated by the Rotation of the Plane BME about the Axis DB, D B, is equal to $\frac{Periph.}{Rad.} \times GDX$, as you find it in *n.* 11. of the *twelfth Lecture.*

PROB. VI.

Fig 185. Let there be any given Figure A D B, whoſe Axis is A D, and Baſe D B, to find the Curve K Z L of ſuch a Nature, that drawing the right Line Z P M any how parallel to D B cutting the ſaid Lines as you ſee, and ſuppoſing the right Line Z T to touch the Curve, the intercepted Line T P may be equal to P M.

In Order to this. Let the Curve O Y Y be of ſuch a Nature, that taking at Pleaſure any Line R, and continuing out P M Y, let P M : R :: R : P Y. Alſo taking D L at pleaſure in B D continu'd out, let D L : R :: R : L E; and with the *Aſymtotes* D L, D G, deſcribe an *Hyperbola* E X X thro' the Point E. Then let the Space L E X H be equal to the Space D O Y P, and let the Lines X H, Y P (continu'd), meet in the Point Z; and Z will be in the Curve required. Which if Z T touches it, we ſhall have T P = P M.

Fig. 186. If the Figure propoſed be a right-angled *Parallelogram* A D B C; the Curve K Z L will be of ſuch a Nature, that D H

will be a *geometrical mean Proportional* between DL, DO of the ſame Order with DP an *arithmetical Mean* between DA and *o* or nothing. And if the Curve KZL be drawn according to this Property and the Tangent ZT *mechanically* determined; the Quadrature of the *hyperbolick* Space LEHX will thereby be had, this being equal to the Rectangle under TP, AP. Hence we may obſerve, 1. That the Spat. $ADLK = R \times DL - DO$. 2. That the Sum of $ZPq = R \times \frac{DLq - DOq}{2}$. And the Sum of ZP Cub. $= R \times \frac{DL\,\text{Cub.} - DO\,\text{Cub.}}{3}$ *&c.* 3. If F be the Cetre of Gravity of the Figure ADLK, and you draw FN perpendicular to AD, and FQ perpendicular to DL; FN will be $= \frac{DL + DO}{4}$, and $FQ = R - \frac{AD \times DO}{LO}$.

PROB. VII.

Let BDH be a right Angle, and BF parallel to DH, and FXG be an *Hyperbola* deſcribed thro' F to the *Aſymtotes* DB, DH. Let a Circle KZL be deſ- Fig. 187.

cribed about the Centre D, and let the Curve AMB be of ſuch a Nature, that taking the Point M any where in it, and drawing the right Line DMZ thro' it, and then taking DI = DM, and drawing IX parallel to BF, let the *Hyperbolick* Space BFXI be equal to twice the *circular Sector* ZDK: to determine the Tangent of the Curve AMB at the Point M.

Draw DS perpendicular to DM, and let DB × BF = R*q*, and make DK : R :: R : P. And DK : P :: DM : DT. Then if TM be drawn, it will touch the Curve AMB.

Obſerv. The Nature of the Curve AMB is ſuch, that DI is a geometrical mean Proportional between DB and DO (or DA) of the ſame Order, with the Arch KZ an arithmetical Mean between *o*, or nothing and the Arch KL. That is, if DI be a Number in a Series of geometrical Proportionals beginning at DB and ending at DA, and *o*, KL be the Logarithms of DB, DA; KZ will be the Logarithm of DI. Or in a Retrograde Way, (as the common Logarithms go on) if DI be a Number ariſing from DO in a geometrical Series, and ending at DB, and *o* be the Logarithm of DO. And the Arch LK that of DB: The Arch LZ will be the Logarithm of DI.

And if the Curve AMB be actually deſcribed, and its Tangent mechanically determined: We may thence get a Circle equal to an *Hyperbolick Space* or *vice verſa*. The Nature of this *Spiral*, as alſo its Menſuration, together with that of the Space BDA, is very well proſecuted by the Ingenious Dr. *Wallis*, in his Book of the *Cycloid*. For which Reaſon I ſhall forbear ſaying any more about it.

PROB. VIII.

Let there be any Space EDG, contained under the Lines DE, DG and the Line ENG. And let R be a given right Line. It is required to find the Curve AMB of ſuch a Nature, that if a right Line DNM be any how drawn from D, and DT be perpendicular to it, and MT touches the Curve AMB; it may be DT : DM :: R : DN. Fig. 188.

Let the Curve KZL be of ſuch a Nature, that $DZ = \sqrt{R \times DN}$, and taking the right Line DB at pleaſure, let DB : R :: R : BF. Alſo let BF, and DH be perpendicular to DB. Then let the *Hyperbola* FXX included within the Angle BDH, paſs thro' the Points F; and let the Space BFXI (ſuppoſing IX parallel to BF) be equal to twice the Space

ZDL. Laſtly, let DM = DI; and the Point M will be in the Curve ſought, which if the right Line TM touches, it ſhall be TD : DM :: R : DN.

PROB. IX.

Fig. 188. Let there be again a Space EDG, as in our laſt; to find the Curve AMB, to which if a right Line DNM be drawn, and DT perpendicular to the ſame, and MT touches the Curve AMB, it may be DT = DN.

Take any Line R, and let $DZq = \frac{R^3}{DN}$. Alſo taking DB to which DH, $BF = \frac{R^3}{DBq}$ are perpendicular, and thro' F, within the *Aſymtotes* DB, DH deſcribe an *Hyperboliform* Figure of the ſecond Order, that is, one whoſe Ordinates BF, or XI be fourth Proportionals in the Ratio of DB to R, or DG to R. Then if the Space BIXF be taken equal to twice the Space ZDL, and DM = DI: the Point M will be in the Curve ſought. Which if MT touches, we ſhall have DT = DN.

PROB.

PROB. X.

Let there be any Figure A D B, whoſe Axis is A D, and Baſe D B, and any how drawing P M parallel to D B, let the Space A P M be any how given or expreſſed. It is required from hence to determine the Ordinate P M. Fig. 189.

Taking any Space R, let R × P Z = A P M. From hence will ariſe the Line A Z Z K, to which find Z O perpendicular. Then it will be P Z : P O :: R : P M.

Examp. Let us call A P, x, and let A P M $= \sqrt{r x^3}$. Therefore P Z $= \sqrt{\frac{x^3}{r}}$: and ſo we ſhall find P O $= \frac{3xx}{2r}$.

And it is $\sqrt{\frac{x^3}{r}} : \frac{3xx}{2r} :: r : \frac{3}{2}\sqrt{rx}$ = P M. Whence A M B is a *Parabola*, whoſe *Parameter* is $\frac{9}{4} r$.

Otherwiſe. Make P Z $= \sqrt{2 \text{APM}}$, and let Z O be perpendicular to the Curve A Z K. Then will P M = P O.

Exam. Let A P = x and A P M $= \frac{x^3}{r}$. Then will P Z $= \sqrt{\frac{2x^3}{r}}$. Conſequently P O will be $= \frac{2xx}{r}$ = P M.

And ſo again the Curve AMB will be a *Parabola.*

PROB. XI.

Fig. 190. Let ADB be any Figure, contained under the right Lines DA, DB, and the Line AMB, and drawing the right Line DM any how from D, let ADM be a given Space; to determine the right Line DM.

Taking any right Line R, let $DZ = \frac{2ADM}{R}$, and let ZO be perpendicular to the Curve AZK, which DH perpendicular to DZ meets; then will $DM = \sqrt{R \times DO}$.

Otherwiſe. Let $DZ = \sqrt{4ADM}$, and ZO perpendicular to the Curve AZK, which DH perpendicular to DZ meets: Then will $DM = \sqrt{DZ \times DO}$.

The excellent *Geometrician*, Dr. *Gregory* of *Aberdeen*, has given us very fine Speculations on *involute* and *evolute* Figures. And indeed, tho' I have no Inclination to ſtrike my Sickle into the Harveſt of another, yet I may be allow'd the Liberty of annexing to theſe *Problems*, one or two Obſervations any-wiſe appertaining to the ſame, that do occur to me, while I am otherwiſe employ'd.

PROB.

PROB. XII.

Let there be any given Figure A D B, whoſe Axis is A D, and Baſe DB; to deſcribe the involute Figure anſwering to the ſame. Fig. 191.

Deſcribe a Circle LXX, about the Centre C, with any Diſtance CL, and let the Curve KZZ be of ſuch a Nature that any how drawing the right Line MPZ parallel to BD, the Rectangle under PM, PZ be equal to the Square of CL (or $PZ = \frac{CLq}{PM}$). Then let the Arch $LX = \frac{\text{Space } DKZP}{CL}$, or the Sector LCX one half the Space DKZP, and in CX take $Cm = PM$. Then will the Line $b\,m\,m$ be the Involute of BMA, or the Space $C\,m\,b$ of the Space ADB. Fig. 192.

Exam. Let AD B be a Quadrant of a Circle. Then (from what is before demonſtrated) the Space DKZP (2 Sect. LCK) : Sect. BDM :: $CLq : DBq$. Whence the Arch LX : Arch BM :: CL : DB. Wherefore the Angle LCX = Angle BDM = Angle DMP. Therefore the Angle Cmb is a right Angle, and ſo the Line $b\,m$ C is a *Semicircle*.

Fig. 193. *Cor.* 1. If two Figures ADB, ADG be analagous, and their *Involutes* be C*mb*, C*nc*, and it be C*m* : C*n* :: DB : DG : it ſhall be reciprocally the Ang. *b*C*m* : *b*C*n* :: DG : DB.

Fig. 194. 2. This may alſo be inverted.

3. Let the Curves C*nc*, CS*b* be analogous in their kind, that is, that the right Line C*n*S any how drawn from C, always has the ſame Ratio. Theſe ſhall be *Involutes* of like Lines.

PROB. XIII.

Fig. 195. Let *b*C*n* be any Figure comprehended under the right Lines C*b*, C*n*, and another Line *bn*. To find the *Evolute* anſwering thereto.

Fig. 196. About the Centre C, deſcribe any how the circular Arch LE, conſtituting the Sector LCE with the right Lines C*b*, C*n*. And drawing CK perpendicular to LC, let the Curve *b*YH reſpecting the right Line CK be ſuch, that any how drawing the right Line C*m*Z, and taking CO = Arch LZ, and drawing OY perpendicular to CK; OY be = C*m*. Then let the Curve BMF be ſo related to the right Line DA that when DP = $\frac{\textit{Space } CbYO}{CL}$, and PM be drawn perpendicular to D A, we have P M = C *m*. And the Space D B F A will be the *Evolute* of C *b n*.

Examp. Let L Z E be the Arch of a Circle deſcribed about the Centre C, and *b m* C be a *Spiral* of ſuch a Nature, that drawing at pleaſure the right Line C *m* Z the Arch E Z may have a given Ratio to the right Line C *m*, ſuppoſe as R to S. It is manifeſt that *b* Y H is a right Line. Becauſe it is always E Z (K O) : C *m* (O Y) :: R : S. Therefore the Evolute B M F is a *Parabola.* Becauſe the Parts A P, A D of the Axis are one to another as the Spaces K O Y, K C *b*, that is, as the Squares of O Y, C *b*, or of P M, D B. Fig. 197. Fig. 198.

Coroll. Theor. I.

If a *cylindrical* Figure be ſet to the Figure *b* C *n* having its Altitude equal to the Periphery of the whole Circle, whereof C L is a radius; this ſame *cylindrical* Figure ſhall be equal to a *Solid* ariſing from the Generation of the Figure C *b* H K about the Axis C K. Fig 195.

Theor.

Theor. II.

Fig. 199. Let there be any Curve AMB, whoſe Axis is AD and Baſe DB, and let the Curve AZL be ſuch, that any how drawing the right Line ZPM, it be $ZP = \sqrt{2APM}$. Alſo let the Curve OYY be ſuch that if the right Line ZP continu'd meets it in Y, and taking any right Line R, it be ZP*q* : R*q* :: PM : PY. And laſtly, let DL : R :: R : LE. Then if an *Hyperbola* EXX be deſcribed thro' E within the Angle LDG, and the right Line ZHK parallel to AD meets the ſame. The Space PDOY ſhall be equal to the *Hyperbolick Space* LHXE.

Hence the *Sum* of all $\frac{PM}{APM} =$

$$\frac{2LEXH}{Rq}.$$

Theor. III.

Fig. 200. Let there be any Curve AMB, whoſe Axis is AD, and Baſe DB. And let the Curve KZL be ſuch, that taking any right Line R, and any how drawing the right Line ZPM parallel to BD, it be $\sqrt{APM}$: PM :: R : PZ. Then will the Space

ADLK be equal to the *Rectangle* under $R \times 2\sqrt{ADB}$. Or $\frac{ADLK}{2R} = \sqrt{ADB}$.

Exam. Let ADB be a Quadrant of a Circle. Then ſhall the Sum of all $\frac{PM}{\sqrt{APM}}$ be $= \sqrt{2DA} \times$ Arch AB.

Theor. IV.

Let there be any Curve AMB, whoſe Axis is AD, and Baſe DB, and let there be two Lines EXK, GYL ſo related, that taking any Point M in the Curve AMB, and drawing the right Lines MPX, MXQ parallel to BD, AD, and ſuppoſing MT to touch the Curve, let TP : PM :: QY : PX. Then ſhall the Figures ADKE, DBLG be equal to one another. Fig. 201.

The Inverſe of this is true too. That is, if the Figures ADKE, DBLG be equal to one another, and MT touches the Curve AMB, it ſhall be TP : PM :: QY : PX.

Obſerv. This *Theorem* is the moſt Fertile of all the *Propoſitions* foregoing. The greater Part of them being either contained in it, or deduced eaſily from it. For ſuppoſing the Line AMB to be of an indeterminate Nature, and either EXK, or GYL to be any how determinate; from thence will reſult a *Theorem* alike to many of thoſe before laid down. For Example, if GYL be a right Line making half a right Angle with BD, (in which Caſe the Points D, G are conceived to coincide) from thence will ariſe the firſt of the *Eleventh Lecture.* And if GYL be parallel to DB, there will be had the 11*th* of the *ſame Lecture.*

Fig. 102. Again, if PM = PX, or (the Lines AMB, or EXK be the ſame) the *tenth* of that *Lecture* will be had. From hence alſo it is evident that taking any Space; infinite Spaces of other Kinds may eaſily be laid down equal to it. As if the Space DLGB be ſuppoſed a Quadrant of a Circle, whoſe Centre is D, and the Curve AMB be a *Parabola,* whoſe Axis is AD; the Curve EXK will ariſe having this Property, that calling DB, r; AP, x; PX, y; and k or $\frac{DBq}{2AD}$ be the *Semi-Parameter,* we ſhall have $\frac{rrk}{2} = kkx + xyy$. But if AMB be an *Hyperbola,* the Curve EXK ariſing will be of another Nature. Here I cannot but accuſe my Foreſight, for not having firſt laid down this *Theorem* (as likewiſe thoſe

following, of almoſt the like Nature and uſe) and then have deduced the reſt from it, (as well as from the others but now to be ſubjoyn'd) which I obſerve may be done. But I have often made it appear in theſe *Lectures*, that *Phrygian Wiſdom* of this kind is very familiar to me.

Theor. V.

Let there be any Space ADB, contain'd under the right Lines DA, DB and the Curve AMB: Let likewiſe the Curves EXK, GYL be ſo related, that if the Point M be taken at Pleaſure in the Curve AMB, and DMX be drawn, DQ be = DM, QY be drawn perpendicular to DB, and DT perpendicular to DM, the right Line MT may touch the Curve AMB. I ſay when theſe Conditions are fulfilled, if TD : DM :: DM × QY : DX *q*: The Space DGLB ſhall be the Double of the Space EDK. Fig 203.

Theor. VI.

Again, let there be any Curve AMB, whoſe Axis is AD, and Baſe DB, and let the Curves EXK, HZO be ſo related to one another and the Axes AD, *a b*, that taking the Point M at pleaſure in the Curve AMB, drawing MPX perpendicular to Fig. 204.

AD, aſſuming $am =$ Arch AM, drawing mZ perpendicular to ab, and ſuppoſing the right Line TM to touch the Curve AMB; it may be TP : TM :: mZ : PX: then ſhall the Spaces ADKE, $abOH$ be equal to one another.

Theor. VII.

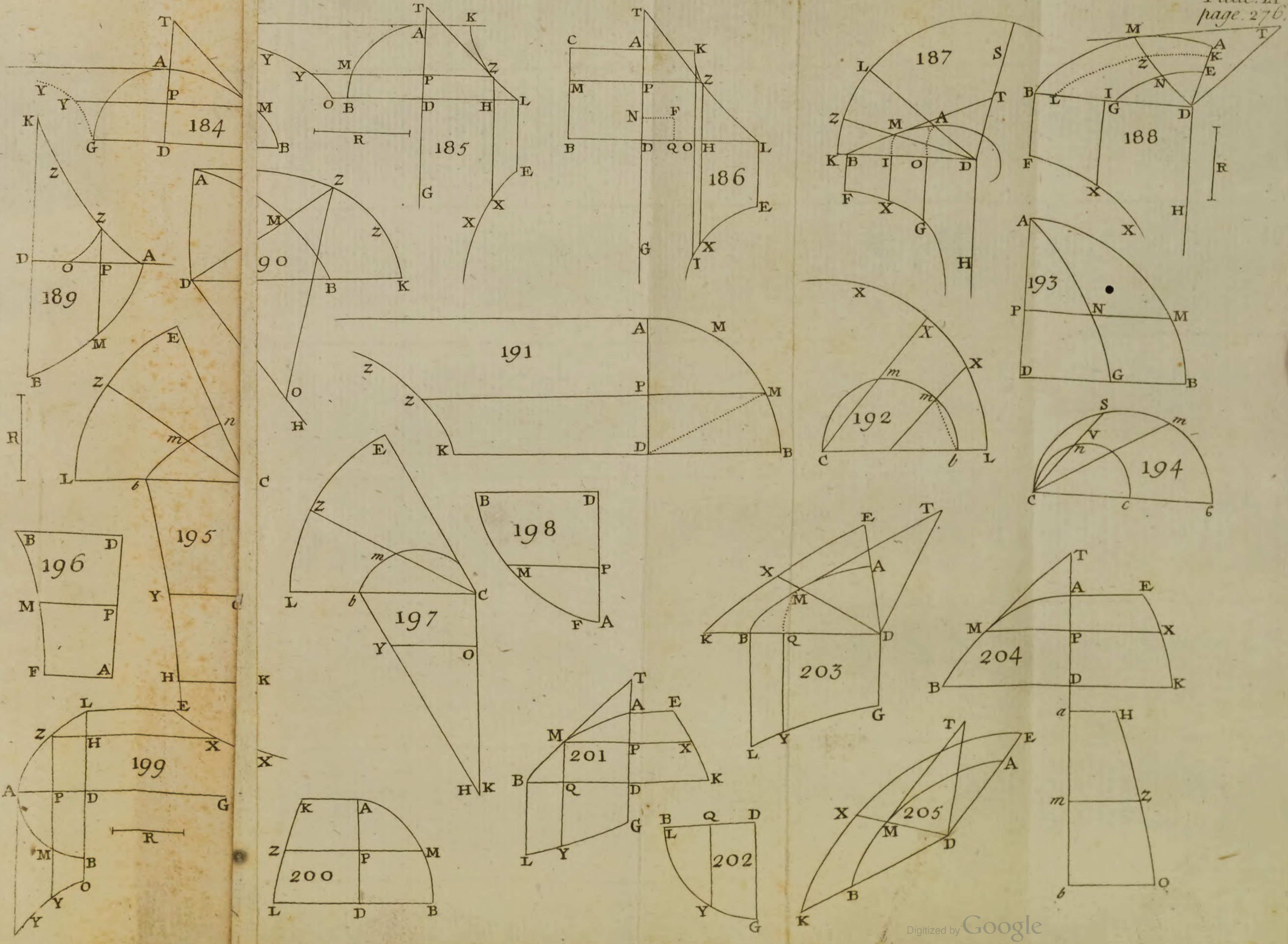
Fig. 204. 205. Let there be any Space ADB, contained under the right Lines DA, DB, and the Curve AMB. Alſo let the Curves EXK, HZO be ſuch, that taking the Point M in the Curve AMB, drawing the right Line DMX, taking $am =$ Arch AM, drawing mZ perpendicular to the right Line ab, and drawing DT perpendicular to DM, and ſuppoſing the right Line MT to touch the Curve AMB, it may be TD : TM :: DM × mZ : DXq. Then ſhall the Space $abOH$ be the Double of the Space EDK.

But let us leave off here.

* * *

LECTURE

Plate IX
page. 276.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LECTURE XIII.

VIETA has ſhewn the Nature of *Equations* from the *Analogy* of the Terms, and *Deſcartes* more clear by drawing or multiplying them into one another. Now I ſhall endeavour to do the ſame in ſome Meaſure, from the Deſcription of Lines accommodated to each, which ſeems certainly to be a Means of ſetting their Nature in a good Light, and expoſing it to View. Come on then.

Obſerve, the *Equations* following are always brought to the ſame *Series*, having the ſame *Coefficients.*

The firſt Series of Equations.

$$a + b = n.$$
$$aa + ba = nn.$$
$$a^3 + baa = n^3$$
$$a^4 + ba^3 = n^4, \&c.$$

Take the right Line B A equal to the Co-efficient *b*, continue it out indefinitely towards H, let the Angles R A H, S B H, be ſemi-right Angles, and let the Lines A L L, A M M, A N N be ſuch, that drawing the right Line G K any how perpendicular to A H (orderly cutting the ſaid Lines in the Points L, M, N, and the right Lines B S, A R in the Points K, Z.) G L * be a *Medial*, G M a *Bimedial*, G N a *Trimedial* between G Z and G K; theſe Lines will ſerve to explain the Nature of the propoſed *Equations*. For if A G (or G Z) be call'd A; then will BG (or G K) $= b + a$. And G L $q =$ $aa + ba$; and G M Cub. $= a^3 + baa$; and G N $qq = a^4 + ba^3$.

Fig. 206.

*See n. 15. Lect. 11.

But obſerve,

1. Drawing A D perpendicular to B H. If on it you take A E $= n$; and draw F E parallel to A H; the Interſection of this with the Lines laid down will exhibit reſpectively the Roots of the propoſed *Equations*; as FK, or EL, or EM, or EN is equal to *a*: That is, equal to the ſeveral Lines A G, by conceiving Perpendiculars

drawn to AH from every Interſection, which will determine the Points G.

2. The more remote the Point G is from the Point A (and indeed GA may be taken greater than any propoſed Magnitude) the more do the Ordinates GK, GL, GM, GN increaſe too ſo that let AE be of what Magnitude ſoever; the Parallel EF will meet the Curve; and conſequently every one of thoſe *Equations* will have one true Root, and but one only; becauſe EF cuts theſe Curves in but one Point.

3. The *Curve* ALL *is an equilateral Hyperbola*, whoſe Axis is AB. The others AMM, ANN are *Hyperboliform Curves.*

4. If AO be $\frac{1}{2}$ AB; and AP $= \frac{1}{3}$ AB, and AQ $= \frac{1}{4}$ AB, and OT, PV, QX be drawn parallel to BS; theſe ſhall be the *Aſymtotes* of the Curves ALL, AMM, ANN.

5. Hence it is manifeſt, in the ſecond Degree, that we ſhall have $a \sqsubset n - \frac{b}{2}$; in the third, $a \sqsubset n - \frac{b}{3}$, in the fourth, $a \sqsubset n - \frac{b}{4}$, which inequalities nevertheleſs, if AE be moderately big, will be but ſmall.

6. There is no *Maximum* or *Minimum* agreeing to theſe *Equations.*

Second Series.

$$a - b = n.$$
$$aa - ba = nn.$$
$$a^3 - baa = n^3.$$
$$a^4 - ba^3 = n^4, \&c.$$

Fig. 207. Let again AB be $= b$; and ſuppoſe AB to be continu'd out towards I, and let the Angles RAI, SBI be half right Angles; then conceive the Curves BLL, BMM, BNN, to be ſuch, that if GZ be drawn any how perpendicular to AI cutting the ſaid Lines in the Points K, L, M, N, Z, (as you ſee) and GL be a *Medial*, GM a *Bimedial*, GN a *Trimedial*, between GZ, GK; theſe Curves will explain the propoſed *Equations.* For if AG (or GZ) be call'd a, it will be; BG (or GK) $= a - b$; and GL$q = aa - ba$; and GM Cub. $= a^3 - baa$; and GN$qq = a^4 - ba^3$.

Not.

1. Drawing AD perpendicular to AI, and EF parallel to AI, if AE be put equal to n : EK, EL, EM, EN will be the reſpective Roots of the *Equations,* or equal to the ſeveral a^s.

2. Becauſe the Ordinates G K, G L, G M, G N do infinitely encreaſe from B towards I, we ſhall always have one true Root, and that but one.

3. The Curve B L L is an *equilateral Hyperbola*, whoſe Axis is A B; and the others are *Hyperboliform Curves.*

4. If A B be Biſected in O, Triſected in P, Quadriſected in Q, and the right Lines O T, P V, Q X be drawn parallel to A R; theſe will be the *Aſymtotes* of the Curves B L L, B M M, B N N.

5. Hence in the ſecond Degree, it follows that $a \sqsubset n + \frac{b}{2}$; in the third $a \sqsubset n + \frac{b}{3}$; in the fourth $a \sqsubset n + \frac{b}{4}$; and if n be big enough, theſe inequalities nearly approach to an equality.

6. Of the true Roots in theſe there is one *Minimum*, *viz.* A B or b.

The Third Series.

$$b - a = n.$$
$$ba - aa = nn.$$
$$ba - a^3 = n^3.$$
$$ba^3 - a^4 = n^4, \text{ \&c.}$$

Let A B be equal to b; and the Angles R A B, S B A half right Angles; and let the Curves A L B, A M B, A N B be ſuch that drawing any how the right Line G K perpendicular to A B (cutting the ſaid Lines as Fig. 208.

as you ſee) and GL be a *Medial*, GM a *Bimedial*, GN a *Trimedial* between AG (or GZ) and GK, theſe Lines will explain the propoſed *Equations.* For putting $AG = a$; it will be $GK = b - a$; and $GLq = ba - aa$; and $GMq = baa - a^3$, and $GNq = ba^3 - a^4$.

Not.

Fig. 208. 1. If on AD (perpendicular to AB) be taken $AE = n$, and EF be drawn perpendicular to AB; the Interſections of this with the Curves will expreſs the Roots a reſpectively.

2. Becauſe the Ordinates to theſe Curves are always of a finite Length, and ſome one of them is a *Maximum*; the *Equations* of this *Series*, according to the Module of the aſſum'd Quantity AE *(n)*, ſometimes have two true Roots *(viz.* when AE ſhall be reſpectively the greateſt or leaſt Ordinate, that is, when EF meets the Curve twice.) Sometimes only one, which is, when AE becomes the *Maximum*, and ſo EF touches the Curve. And ſometimes none, *viz.* when AE exceeds the *Maximum*; and ſo EF does no where meet the Curve.

3. In the ſecond Degree, if AO be = OB, and OT be an Ordinate; OT will be the *Maximum*; and ſo one of the

Roots will be greater than $\frac{AB}{2}$, and the other leſs. In the third, if A P be $=$ 2 P B, and the Ordinate P V drawn; P V will be the *Maximum.* Whence one of the Roots will be greater than $\frac{1}{3}$ A B, and the other leſs. Laſtly, in the fourth Degree, if A Q $=$ 3 Q B, and Q X be made an Ordinate; Q X will be the *Maximum.* Therefore one of the Roots will always be greater than $\frac{1}{4}$ A B, and the other leſs.

4. Hence it follows, that in the ſecond Degree, if $n \sqsubset \frac{b}{2}$, in the third $n^2 \sqsubset \frac{4b^3}{9} - \frac{8b^3}{27} = \frac{4b^3}{27}$, in the fourth $n^4 \sqsubset \frac{27}{64}b^4 - \frac{81}{256}b^4 = \frac{27b^4}{256}$, there can be no Root had.

5. Of all the Roots A B or b is the *Maximum.*

6. The Point T is the common *Interſecti-on* or *Nodus* of all the Curves, and if n be $= \frac{b}{2}$; A O or $\frac{b}{2}$ will be always one Root.

7. The Curve A L B is a *Circle,* and the others A M B, A N B, have ſome ſort of relation to it.

1.	2.	3.
$a+b=n.$	$a-b=n.$	$b-a=n.$
$a+b=\frac{nn}{a}.$	$a-b=\frac{nn}{a}.$	$b-a=\frac{nn}{a}.$
$a+b=\frac{n^3}{aa}.$	$a-b=\frac{n^3}{aa}.$	$b-a=\frac{n^3}{aa}.$
$a+b=\frac{n^4}{a^3}.$	$a-b=\frac{n^4}{a^3}.$	$b-a=\frac{n^4}{a^3}.$

The *Equations* aforegoing may otherwiſe, and perhaps more conveniently be explain'd, by taking only one Line for each Degree of the *three Series*, after the following Manner.

Fig. 209. 210. Let the right Line AH be indefinitely extended, and AD perpendicular to the ſame; on which take AB = *n*, and draw BK parallel to AH. Then let the Lines LXL, MXM, NXN be ſuch that taking any Point G on AH, and drawing GK parallel to AD; in the Proportion of AG to GK (or AB) let GL be a *third Proportion*, GM a *fourth*, and GN *a fifth*; theſe Curves will ſerve to explain the Nature of the propoſed Equations.

For taking AE = *b*; towards I for the firſt *Series*, and towards H for the ſecond and third; make the Angle FEH half a right one, inclining towards H for the firſt and ſecond *Series*, and reclining therefrom, for the third, as the Figure ſufficiently ſhews. Then will the reſpective Interſections of the right Line E F and the ſaid Curves determine the Roots *a*. For if the Perpendiculars L G, M G, N G be conceived to be drawn; the intercepted Parts A G will be reſpectively equal to the Roots *a*.

Not.

From hence it is manifeſt, that

1. In this Explanation the *Co-efficient* b is determinate, as n was in that aforegoing.

2. In the firſt and ſecond *Series*, there is always one poſitive Root, and that one only.

3. In the ſecond *Series* the leaſt Root is equal to A B or n.

4. The common *Nodus* or *Point* X is where B X (or a) $= n$.

5. In the third *Series* there are ſometimes two poſitive Roots, as when E F twice cuts the Curves; ſometimes one only, when E F touches one of them; (and this happens, in the ſecond Degree, when $a = \frac{b}{2}$; in the third, when $a = \frac{2}{3} b$; in the fourth, when $a = \frac{3}{4} b$); and ſometimes

times none, when EF falls below the Tangents; and ſo no where meets the Curves.

6. The Curve of the ſecond Degree is an *Hyperbola*, the reſt are of the *Hyperbolick* Kind, and the common *Aſymtotes* are the right Lines AH, AD.

Fourth Series.

$$a + \frac{cc}{a} = n.$$

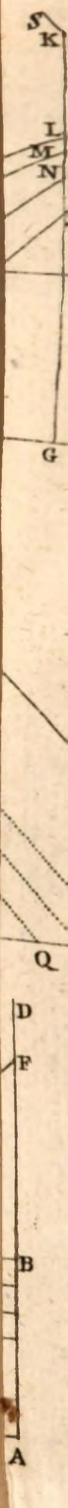
$$aa + cc = nn.$$

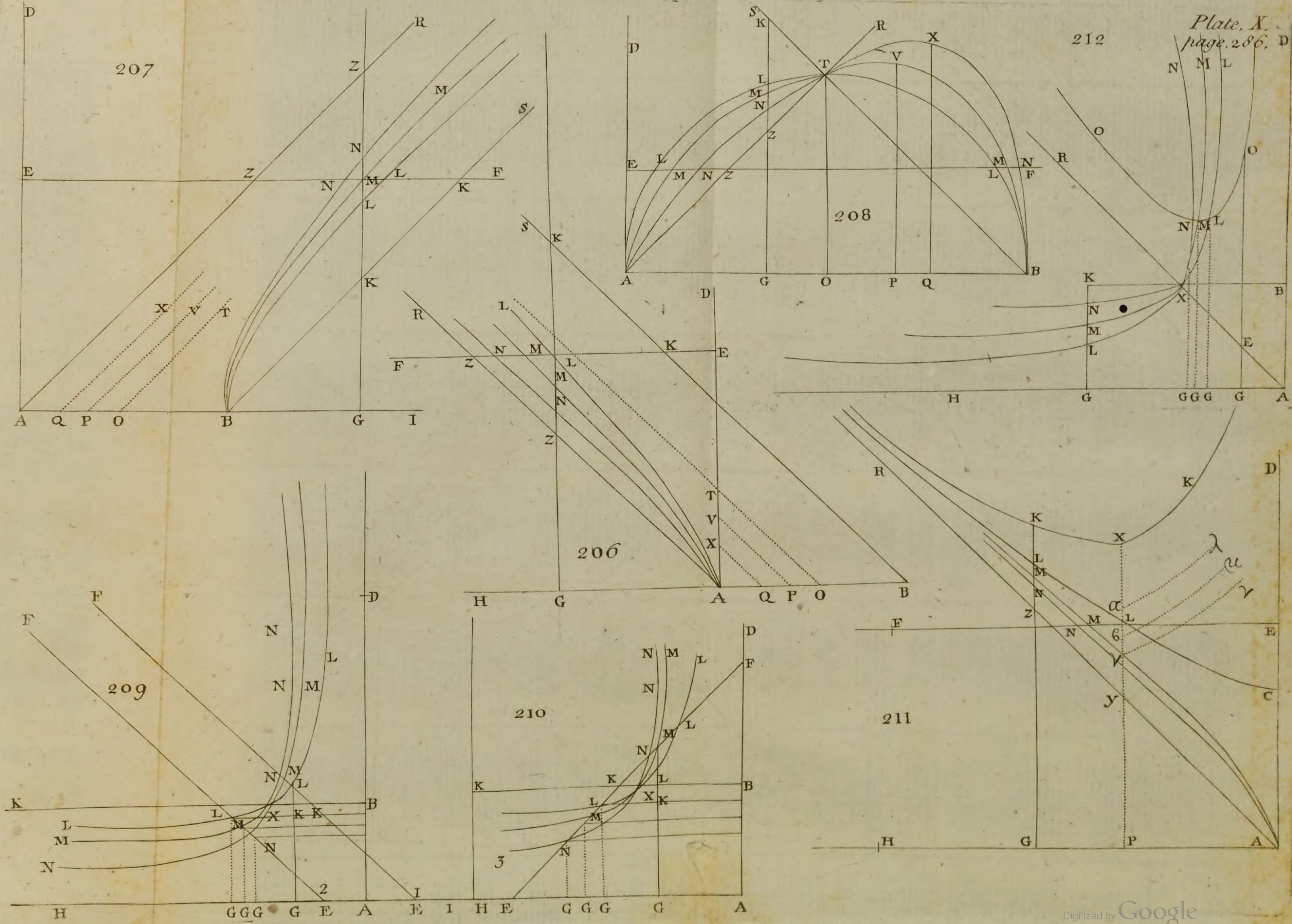
$$a^3 + cca = n^3.$$

$$a^4 + ccaa = n^4.$$

Fig. 211. Let AH be a right Line indefinitely extended, and AD perpendicular to it; make RAH half a right Angle, draw GZK any how parallel to AD, and make AG : AC :: AC : ZK, and deſcribe the *Hyperbola* KXK thro' K within the Angle DAR. Then let the Curves CLL, AMM, ANN be ſuch, that GL be a *Medial*, GM a *Bimedial*, and GN a *Trimedial* between GZ, GK. Theſe Curves will ſerve to explain the Buſineſs. For if AG (or GZ) be call'd a, it will be GK $= a + \frac{cc}{a}$; GL$q = aa + cc$, GM Cub. $= a^3 + cca$, and GN$qq = a^4 + ccaa$.

Not.

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Not.

1. The Roots are expreſs'd, as before, putting $AE = n$, and drawing EF parallel to AH.

2. If AP be $= AC$; PX will be the leaſt of the Ordinates to the *Hyperbola* KXK. Therefore in the firſt Degree, if AE (or n) $\sqsupset PX$, there will be no Root.

3. The Curve CLL is an *equilateral Hyperbola*, whoſe *Centre* is A, and *Semi-Axis* AC; which is the *leaſt* of the Ordinates too. Otherwiſe if $n \sqsubset c$, there will always be one Root, and that but one.

4. The other Curves AMM, ANN are of the *Hyperbolick* Kind running out *ad Infinitum*; whence there will be one true Root, and no more.

5. If $Y\alpha = \frac{1}{2} YX$; $Y\beta = \frac{1}{3} YX$, $Y\gamma = \frac{1}{4} YX$, and thro' the Points α, β, γ, *Hyperbola*'s $\alpha\lambda$, $\beta\mu$, $\gamma\nu$, be conceived to be drawn (having DA, AR for *Aſymtotes)* theſe will be the *Aſymtotes* of the Curves CLL, AMM, ANN. (The Curve Lines afterwards to be deſcribed have alſo ſuch ſort of *Aſymtotes*, tho' we do not mention them.)

6. Hence in the ſecond Degree $a + \frac{cc}{2a} \sqsubset n$, in the third $a + \frac{cc}{3a} \sqsubset n$, in

the fourth $a + \frac{cc}{4a} \sqsubset n$. Which inequality nevertheleſs is ſo much the leſs as AE (or n) is the greater.

$$a + \frac{cc}{a} = n.$$

$$a + \frac{cc}{a} = \frac{nn}{a}.$$

$$a + \frac{cc}{a} = \frac{n^3}{aa}.$$

$$a + \frac{cc}{a} = \frac{n^4}{a^3}.$$

Fig. 212. This Series may be explained according to the ſecond Way aforegoing, and that by exhibiting the ſame Curves LXL, MXM, NXN of ſuch a Nature, that drawing GK any how perpendicular to AH; GL may be $= \frac{nn}{\text{AG}}$, GM $= \frac{n^3}{\text{AG}\,q}$, and GN $= \frac{n^4}{\text{AG}}$ Cub. For if HAR be a half right Angle, and GEO be any how drawn perpendicular to AH, and it be made GE : c :: c : EO; and you deſcribe a *Hyperbola* OO thro' O within the *Aſymtotes* AD, AR; the Interſections of this and the ſaid Curves LXL,

MXM, NXN, will determine the reſpective Roots *a*. For drawing LG, MG, NG perpendicular to AH; the intercepted Parts AG will be reſpectively equal to the ſeveral *a*'s.

After the like manner may all the ſubſequent *Series* be explain'd. But we ſhall continue on the Buſineſs only according to the firſt Way.

Fifth Series.

$$\frac{cc}{a} - a = n.$$

$$cc - aa = nn.$$

$$cca - a3 = n3.$$

$$ccaa - a4 = n4.$$

Sixth Series.

$$a - \frac{cc}{a} = n.$$

$$aa - cc = nn.$$

$$a3 - cca = n3.$$

$$a4 - ccaa = n4.$$

Make the Angle RAI half a right one, Fig 213. and let AD be perpendicular to AI, and make AC $= c$. Then drawing GZ any how parallel to AD, let AG (or GZ) : AC :: AC : ZK, and thro' K deſcribe an *Hyperbola* KYK within the Angle DAR. Then let the Curves CLYH λ, AMYH μ, ANYH ν be ſuch, that GL be a *Medial*, GM a *Bime-*

dial, and GN a *Trimedial* between AG (or GZ) and GK; theſe will ſerve to explain our Buſineſs.

This is evident, as in what foregoes, and how the Roots are reſpectively determined. But we may obſerve further.

Not.

1. The Curves CLH, AMH, ANH, do belong to the *fifth Series*, and the reſt $HL\lambda$, $HM\mu$, $HN\nu$ to the ſixth.

2. As to the Curves appertaining to the *fifth Series*, if $A\phi = \surd\frac{ACq}{2}$, and you draw the Ordinate ϕY; Y will be the common Interſection or *Nodus* of the Curves.

3. In the firſt Degree of them the Ordinate AK is infinite. In the ſecond AC is the greateſt. In the third, if $AP = \surd\frac{ACq}{3}$, and the Ordinate PV be drawn; this will be the *Maximum*. Whence one of the Roots is always greater than $\surd\frac{ACq}{3}$ and the other leſs. In the fourth, if $AQ = \surd\frac{ACq}{4} = \frac{AC}{2}$, and the Ordinate QX be drawn; QX will be the *Maximum*. Wherefore one of the Roots will be greater and the other leſs than $\frac{AC}{2}$.

4 Conſequently in the ſecond Degree, if $n \sqsubset c$; in the third, $n^3 \sqsubset cc \surd \frac{cc}{3} - \frac{cc}{3} \surd \frac{cc}{3} = \frac{2}{3} cc \surd \frac{cc}{3}$, or $n^6 \sqsubset \frac{4}{27} c^6$; in the fourth $n^4 \sqsubset \frac{c^4}{4} - \frac{c^4}{16} = \frac{3}{16} c^4$; then there will be no Root. For in theſe Caſes the right Line E F falls above the Curves; and ſo does not meet them.

5. In all theſe the greateſt poſſible Root is $AH = AC$.

6. The Curve C Y H, is a *Quadrant* of a Circle, and the reſt A M H, A N H are of the *Circle-Kind.*

7. The Curve H L L belonging to the *ſixth Series* is an *equilateral Hyperbola,* whoſe Axis is A H. And the reſt are of the *Hyperbolick Kind.* Therefore what elſe regard this *Series* are manifeſt.

Seventh Series.

Fig. 214.

$$a + b + \frac{cc}{a} = n.$$

$$aa + ba + cc = nn.$$

$$a^3 + baa + cca = n^3.$$

$$a^4 + ba^3 + ccaa = n^4, \&c.$$

Fig. 214.

On the right Line B A H indefinitely extended aſſume $AB = b$, and on A D perpendicular to B H take $AC = c$. Alſo let the Angles H A R, H B S be half right ones, and drawing at pleaſure G Y perpendicular to A H cutting B S in Y, make A G : A C :: A C : Y K; and thro' K within the Angle D V S deſcribe the *Hyperbola* K K K. Then let the Curves C L L, A M M, A N N be of ſuch a Nature that G L be a *Medial*, G M a *Bimedial*, and G N a *Trimedial* between A G (or G Z) and G K: Theſe will accompliſh the Buſineſs. For G K is $= a + b + \frac{cc}{a}$; $GLq = aa + ba + cc$, $GM^3 = a^3 + baa + cca$, and $GNqq = a^4 + ba^3 + ccaa$.

Obſerv.

1. The Curve C L L of the ſecond Degree is part of an *equilateral Hyperbola*, whoſe Centre is O, biſſecting A B, and if we have A C $\sqsubset$ A O; then is O H (perpendicular to AB) $= \sqrt{ACq - AOq}$ its *Semi-Axis*. And if A C $\sqsupset$ A O, its Axis is $OI = \sqrt{AOq - ACq}$. The reſt of the Curves A M M, A N N are of the *Hyperbolick Kind.*

2. Hence in the ſecond Degree, if $n \sqsupset c$, there will be no true Root. Otherwiſe in all there will be one, and but one; becauſe the right Line E F will once and no oftner cut the Curves.

The Eighth Series.

$$\frac{cc}{a} + b - a = n.$$

$$cc + ba - aa = nn.$$

$$cca + baa - a^3 = n^3.$$

$$ccaa + ba^3 - a^4 = n^4, \&c.$$

The

The Ninth Series.

$$a - b - \frac{cc}{a} = n.$$

$$aa - ba - cc = nn.$$

$$a^3 - baa - cca = n^3.$$

$$a^4 - ba^3 - ccaa = n^4, \&c.$$

On the right Line AI take $AB = b$, and on AD perpendicular to AI aſſume $AC = c$, make the Angles IAR, ABS half right Angles, and draw the right Line ZGK any how perpendicular to AI, cutting BS at ξ, and let $AG : AC :: AC : \xi K$. Then deſcribe an *Hyperbola* KYHK thro' K within the Angle DSB. And let the Curves CLHLλ, AMHMμ, ANHNν be ſuch, that GL be a *Medial*, GM a *Bimedial*, and GN a *Trimedial* between AG and GK: theſe Curves will ſerve our purpoſe; as will appear from what foregoes. Fig. 215.

Obſerv.

1. The Curves CLH, AMH, ANH do belong to the *eighth Series*, and the reſt HLλ, HMμ, HNν to the *ninth*.

2. With regard to the *eighth Series*, if AB be biſſected in O, and the Ordinate OT be drawn to the Curve CLH; OT is the *Maximum*. But if you make $AP = \frac{b}{3} + \sqrt{\frac{bb}{9} + \frac{cc}{3}}$, and PV be an Ordinate to the Curve AMH; PV will be

the *Maximum*. Alſo if $AQ = \frac{3}{8}b + \sqrt{\frac{9}{64}bb + \frac{cc}{2}}$, and QX be an Ordinate to the Curve ANH; QX will be the *Maximum*.

3. Hence in the ſecond Degree, if $n \sqsubset \sqrt{cc + \frac{bb}{4}}$; in the third, $n^3 \sqsubset ccf + bff - f^3$, ſuppoſing $f = \frac{b}{3} + \sqrt{\frac{bb}{9} + \frac{cc}{3}}$; in the fourth, $n^4 \sqsubset ccgg + bg^3 - g^4$, admitting $g = \frac{3}{8}b + \sqrt{\frac{9}{64}bb + \frac{cc}{2}}$: There will be no true Root. For in theſe Caſes the right Line EF will not meet the Curves.

4. If $A\varphi = \frac{b}{4} + \sqrt{\frac{bb}{16} + \frac{cc}{2}}$, and φH be an Ordinate; Y will be the *Nodus* of the Curves. And ſo if $n = A\varphi$; $A\varphi$ will be one of the Roots in all.

5. The Curve CLH is the *Circumference* of a *Circle*, whoſe Centre is O; the reſt AMH, ANH being of the *Circle-Kind*.

6. In the ſecond Degree, and this only, if $n = c$, there is only one Root.

7. The greateſt of the Roots in this, which is alſo the leaſt in the *ninth Series*, is $AH = \frac{b}{2} + \sqrt{\frac{bb}{4} + cc}$.

8. The Curve HLλ is an *equilateral Hyperbola*, whoſe *Semi-Axis* is OH. The reſt HMμ, HNν, are of the *Hyperbolick-Kind*. Conſequently in the *ninth Series*, there is always one Root, and no more.

The Tenth Series.

$$a + b - \frac{cc}{a} = n.$$
$$aa + ba - cc = nn.$$
$$a^3 + baa - cca = n^3.$$
$$a^4 + ba^3 - ccaa = n^4, \&c.$$

The Eleventh Series.

$$\frac{cc}{a} - b - a = n.$$
$$cc - ba - aa = nn.$$
$$cca - baa - a^3 = n^3.$$
$$ccaa - ba^3 - a^4 = n^4, \&c.$$

On the right Line BAH take BA $= b$; Fig. 216. and on AD perpendicular to AH let AC $= c$. And let the Angles HAR, HBS be half right Angles. Then any how drawing the right Line GKξ perpendicular to AH, cutting BS in ξ, let AG : AC :: AC : ξK, and thro' K within the *Aſymtotes* VD, VS deſcribe the *Hyperbola* KYHK. Let the Curves CLHLλ, AMHMμ, ANHν be ſuch, that GL be a *Medial*, GM a *Bimedial*, and GN a *Trimedial* between AG (or GZ) and GK.

Theſe will ſerve to explain the propoſed *Series.* Which is evident as in the foregoing ones.

Obſerv.

1. The Curves $HL\lambda$, $HM\mu$, $HN\nu$ do belong to the *tenth Series*, and the reſt CLH, AMH, ANH to the *eleventh.*

2. The Curve $HL\lambda$ is an *equilateral Hyperbola*, and the Curve CLH is a Part of the *Circumference* of *a Circle*, the Centre O, biſſecting AB, being common to both.

Therefore $AH = \sqrt{\frac{bb}{4} + cc} : - \frac{b}{2}$.

3. In the *tenth Series* there is always one Root, and no more. But in the *eleventh*, there is ſometimes two, ſometimes one, and ſometimes no Root.

4. If $A\phi = \frac{cc}{b}$; and $A\psi = \sqrt{\frac{bb}{16} + \frac{cc}{2}} : - \frac{b}{4}$, and you draw the Ordinates ϕY, ψX; the Points Y, X are the *Nodes* of the Curves.

5. In the ſecond Degree of the *eleventh Series* AC is the greateſt Ordinate. But if $AP = \sqrt{\frac{bb}{9} + \frac{cc}{3}} : - \frac{b}{3}$, and from P be drawn the Ordinate $P\gamma$ to the Curve AMH; this will be the *Maximum.*

Alſo if $AQ = \sqrt{\frac{9bb}{64} + \frac{cc}{2}:} - \frac{3b}{8}$, and from Q be drawn the Ordinate $Q\delta$ to the Curve ANH: This will alſo be the *Maximum*. And ſo we are to judge of the Limits of the Roots according to the Obſervations on the *eighth Series.*

The Twelfth Series.

Fig. 217.

$$a - b + \frac{cc}{a} = n.$$
$$aa - ba + cc = nn.$$
$$a^3 - baa + cca = n^3.$$
$$a^4 - ba^3 + ccaa = n^4, \&c.$$

The Thirteenth Series.

$$b - a - \frac{cc}{a} = n.$$
$$ba - aa - cc = nn.$$
$$ba^2 - a^3 - cca = n^3.$$
$$ba^3 - a^4 - ccaa = n^4, \&c.$$

Fig. 217.

For theſe, let $AB = b$, $AC = c$, and ABS the half of a right Angle, and $G\xi$ any how perpendicular to AB, and let $AG : AC :: AC : \xi K$. Then let the *Hyperbola* KHKIK be deſcribed to the *Aſymtotes* SA, SB, and the Curves CLHLIL λ, AMHMIM μ, ANHNIN ν, be ſuch, that GL be a *Medial*, GM a *Bimedial*,

Bimedial, and GN a *Trimedial* between AG, GK.

Obſerv.

1. The Curves CLH, AMH, ANH, and the Curves ILλ, IMμ, INν, do relate to the *twelfth Series*, but the intermediate Curves HLI, HMI, HNI to the *thirteenth Series*.

2. The Curves CLH, ILλ, are *equilateral Hyperbola*'s, whoſe common Centre, biſſecting the right Line AB, is O, and *Semi-Axis* OH (or OI) $= \sqrt{AOq - ACq}$. The reſt are ſuch as the Figure ſhews.

3. The Curve HLLI is a *Semicircle*, and what the reſt are the Scheme ſhews.

4. If $A\zeta = \frac{cc}{b}$, $A\psi = \frac{b}{4} - \sqrt{\frac{bb}{16} - \frac{cc}{2}}$, and $A\varphi = \frac{b}{4} + \sqrt{\frac{bb}{16} - \frac{cc}{2}}$, and you draw the Ordinates ζV, ψX, φY: the Points V, X, and Y are the *Nodes* of the Curves. If $b \sqsupset \sqrt{8cc}$, the *Nodes* X, Y will be wanting, and when $b = \sqrt{8cc}$; theſe do Coincide.

5. The *greateſt* of the Ordinates to the Curve CLH is AC. But if $AP = \frac{b}{3} - \sqrt{\frac{bb}{9} - \frac{cc}{3}}$, and the Ordinate P$\gamma$

be drawn to the Curve AMH: $P\gamma$ will be the *Maximum.* Alſo if AQ $= \frac{3}{8}b - \sqrt{\frac{9}{64}bb - \frac{cc}{2}}$, and the Ordinate $Q\delta$ be drawn to the Curve ANH; $Q\delta$ will be the *Maximum.*

6. The *greateſt* of the Ordinates to the Curve HLLI is OT. But if AP $= \frac{b}{3} + \sqrt{\frac{bb}{9} - \frac{cc}{3}}$, and the Ordinate pg be drawn to the Curve HMI: pg will be the *Maximum.* And if AQ$q = \frac{3}{8}b + \sqrt{\frac{8}{64}bb - \frac{cc}{2}}$, and the Ordinate qd be drawn to the Curve HNI; qd will be the *Maximum.*

7. From hence the Limits of the Roots may be known, as we have obſerved in our Obſervations on the *eighth Series.*

8. In the *twelfth Series* there are ſometimes three, ſometimes two, and ſometimes one Root. But in the *thirteenth* ſometimes two, ſometimes one, and ſometime none.

9. And this is evident ſuppoſing $\frac{b}{2} \sqsubset c$.

But if $\frac{b}{2} = c$; the *thirteenth Series* will vaniſh; the Points H, O, I Coincide, the right Line AB touch the *Hyperbola* KKK, and the Curves C L H, I L λ become right Lines.

10. If $\frac{b}{2} \sqsupset c$, the *thirteenth Series* will alſo vaniſh, the *Hyperbola* K K K falling entirely below the right Line AB, in which Caſe the Curve C L L will be an *equilateral Hyperbola*, having the Centre O, and Fig 218. Semi-Axis O T (perpendicular to *A* B) $= \surd AC q - A O q$. Then the Curves A M M, A N N do run out *ad Infinitum*, ſo that the *Equations* in the 12*th Series*, have always one Root, and never more.

It may be ſufficient to have hinted theſe things, as alſo to have hitherto handled the whole Affair particularly; we ſhall now add a few more general Obſervations, which are,

1. That the *Curve* explaining any propoſed Equation, may be deſcribed after Fig. 219 this Manner. Let, for Example, the *Equation* be $a^5 + b a^4 + c c a^3 - d^3 a^4 - f^4 a = n^5$. Take the Point A on the indefinite right Line HI, for the Bound or Origin of the Roots. Then taking A G at pleaſure, for the indeterminate Root *a*, make G K equal to the firſt Degree of the Series, containing the propoſed Equation, as let G K (be here) $= a + b + \frac{cc}{a} - \frac{d^3}{aa} - \frac{f^4}{a^3}$, (*viz.* the Continuation of the Ratio *a* to *c* once,

is $\frac{cc}{a}$; the Continuation of *a* to *d* twice is $\frac{d^3}{aa}$; and ſo on.) Then let GO be the firſt of as many mean Proportionals between AG,GK, as the Degree of the propoſed Equation requires, which may be known from the pure power of the Root ſought, being in this Caſe four mean Proportionals. And the Curve A O O, drawn thro' Points thus found, will anſwer the purpoſe.

2. We have hitherto not at all touch'd upon negative or falſe Roots. But theſe may be found thus. Inſtead of the Equation propoſed ſubſtitute another whoſe Signs in the odd Places (aſſigning them, even to the empty ones) are contrary to the Signs of the propoſed *Equation*; then will the true or poſitive Roots of this [ſubſtituted] *Equation*, be the falſe or negative Roots of the propoſed *Equation*. For Example, let the Equation be $a^3 + baa = n^3$, or $a^3 = baa * - n^3 = 0$. Subſtitute the *Equation* $a^3 - baa * + n^3 = 0$, and the true Roots of this found, as ſought above in the third Series, will be the *falſe Roots* of the *propoſed Equation*. Again let $a^3 - baa = n^3$. Or $a^3 - baa - n^3 = 0$. Subſtitute the *Equation* $a^3 + baa + n^3 = 0$. This has no true

Root, and therefore the propoſed *Equation* admits of no falſe Root.

3. If any true Root be given, the falſe Roots may be found, by a certain *Equation* of a lower Degree, thus: Let $a^3 + baa = n^3$ be an *Equation* one of whoſe Roots, which call f, is known, and take another exactly like it having the ſame *Co-efficients*, but with this Difference that f ſtands for a, that is, $f^3 + bff = n^3$. Then is $a^3 + baa = n^3 = f^3 + bff$; and ſo $a^3 + baa - f^3 = bff = o$. Divide this *Equation* (which you may always do) by $a - f$ and there comes out

$$aa. \begin{matrix} + ba + bf \\ + fa + ff \end{matrix} = o.$$

The Roots of which *Equation* will be the ſame as the remaining Roots of the *Equation* propoſed; which accordingly will be found to have two falſe Roots; and ſo changing the Signs of the even Places we have

$$aa \begin{matrix} - ba + bf \\ - fa + ff \end{matrix} = o:$$

Therefore the true Roots of this *Equation* will expreſs the falſe ones of that propoſed. And this way moreover ſhews the Nature of an *Equation*; ſo far as it ariſes from the Multiplication of others into one another.

4. If you make $n = o$ in any Degree of the Series, you may find the greateſt and leaſt Roots in the Series. As in the *eighth Series*, if $ba - aa + cc = o$, and

ſo $cc = aa - ba$, the *greateſt Root* will be $a = \frac{b}{2} + \surd\frac{bb}{4} + cc$. So in the *twelfth Series*, if $aa - ba + cc = 0$, and ſo $cc = ba - aa$, the *greateſt Root* will be $a = \frac{b}{2} + \surd\frac{bb}{4} - cc$, and the leaſt *Root* $a = \frac{b}{2} - \surd\frac{bb}{4} - cc$.

5. The *Nodes* or *Interſections* in any Degree of what *Series* ſoever may be found by putting $a = n$, as in the *eighth Series* where $ba - aa + cc = nn$. Let $a = n$, then will $ba - aa + cc = aa$, or $cc = 2aa - ba$, or $\frac{cc}{2} = aa - \frac{ba}{2}$. Conſequently $a = \frac{b}{4} + \surd\frac{bb}{16} + \frac{cc}{4}$. Alſo in the *twelfth Series*, where $aa - ba + cc = n^4 = aa$, it will be $cc = ba$. And ſo $a = \frac{cc}{b}$.

6. The Inveſtigation of the *greateſt* and *leaſt* Ordinates, has been performed in the Notes by ſeveral ways and means. Which notwithſtanding may be determined at once, together with the *Tangents* of the Curves, after the following manner. Let A γ H be a Curve explaining the *eleventh Series*, and of ſuch a Degree that the

Fig. 220.

the *Equation* be $cca - baa - a^3 = n^3$. Suppoſing γT to touch the Curve, and γP an Ordinate to AH, I find, from what is above demonſtrated, that $PT = \frac{3n^3}{3aa + 2ba - cc}$. Then I conſider if the Ordinate Pγ be a *Maximum*, the Tangent will be parallel to AH, or the right Line PT infinite. Therefore ſince $n^3 = PT \times : 3aa + 2ba - cc$, and n is finite, it is evident that $3aa + 2ba - cc = 0$, or $aa + \frac{2}{3}ba = \frac{cc}{3}$. And ſo $\surd \frac{bb}{9} + \frac{cc}{3} : -\frac{b}{3} = a =$ AP.

7. Finally, I obſerve that the Limits of the Roots are derived from the *greateſt* and *leaſt Ordinates*. For if the Value of the Root, as AP in the Example juſt now named, belonging to the greateſt *Ordinate* be found, and it be ſubſtituted for a every where in the *Equation*, if what comes out is leſs than the *Homogene* of *Comparation*, as they call it, the Problem is impoſſible, or certainly wants ſome Roots, which the Degrees and Kinds of the *Equation* do ſhew. The ſame muſt be conceived of the *leaſt Ordinates*; only here the *Sum* ariſing

muſt be greater than the ſaid *Homogeneal* Quantity, whereby ſome one or all the Roots may be had. *Examples* of which may be ſeen in the Premiſſes.

ADDENDA.

FROM the Conſideration of Tangents, I have found out a Method by which ſeveral *Theorems* (not to be deſpiſed) about *Maximums* are moſt expeditiouſly deduced. Some of which I ſhall point out.

Fig 224. Let there be any Curve ALB whoſe Axis is AD, and Baſe DB, and let LG, $\lambda\gamma$, be parallel to it. Alſo let LT touch the Curve.

Theor. I.

Let m be any Number expreſſing Powers. If you put $DG^{\overline{m-1}} \times TG = GL^{m}$; $DG^{\overline{m}} + GL^{\overline{m}}$ will be a *Maximum,*

ximum, or greater than $\overline{D\gamma}^{m} + \overline{\gamma\lambda}^{m}$.

Theor. II.

Alſo aſſuming the Number m, if you put $\overline{BL}^{m-1} \times TL = \overline{GL}^{m}$; $\overline{GL}^{m} + \overline{BL}^{m}$ will be a *Maximum*, or greater than $\gamma\lambda^{m} + B\lambda^{m}$.

Theor. III.

Let m, n, be any Numbers; if you put $m \times TG = n \times DG$; $\overline{DG}^{m} \times \overline{GL}^{n}$ ſhall be a *Maximum*, or greater than $\overline{D\gamma}^{m} \times \gamma\lambda^{n}$.

Theor. IV.

And if you put $m \times TL = n \times$ Arch BL; $\overline{GL}^{n} \times \overline{BL}^{m}$ ſhall be a *Maximum*, or greater than $\overline{\gamma\lambda}^{n} \times \overline{B\lambda}^{m}$.

Theor. V.

If it be $TG \times GL = DGLB$; $DGLB \times GL$ will be a *Maximum*, or greater than $D\gamma\lambda B \times \gamma\lambda$.

Theor. VI.

If $TG \times GL = 2\,DGLB$; $GL \times \surd DGLB$ will be a *Maximum*, or greater than $\gamma\lambda \times \surd D\gamma\lambda B$.

When theſe are demonſtrated, it will be no difficult Buſineſs to find out ſeveral others of the like Nature.

And the following *Theorems* will ſerve to prove them more compendiouſly.

Fig. 225. Let there be two Curves A G B, D H C whoſe common Axis is A D, and Ordinates increaſe from A to D B in an inverſe Situation, and decreaſe from D to A C, and at the Ends of the common Ordinate G E H, let the right Line G S touch the Curve A G B, and the right Line H T the Curve D H C.

1. If the right Line H T be parallel to the right Line G S; G E H will be the greateſt Sum of the Ordinates both lying in the ſame right Line.

For drawing any how O K F L P parallel to G E H (cutting the Lines as *per*

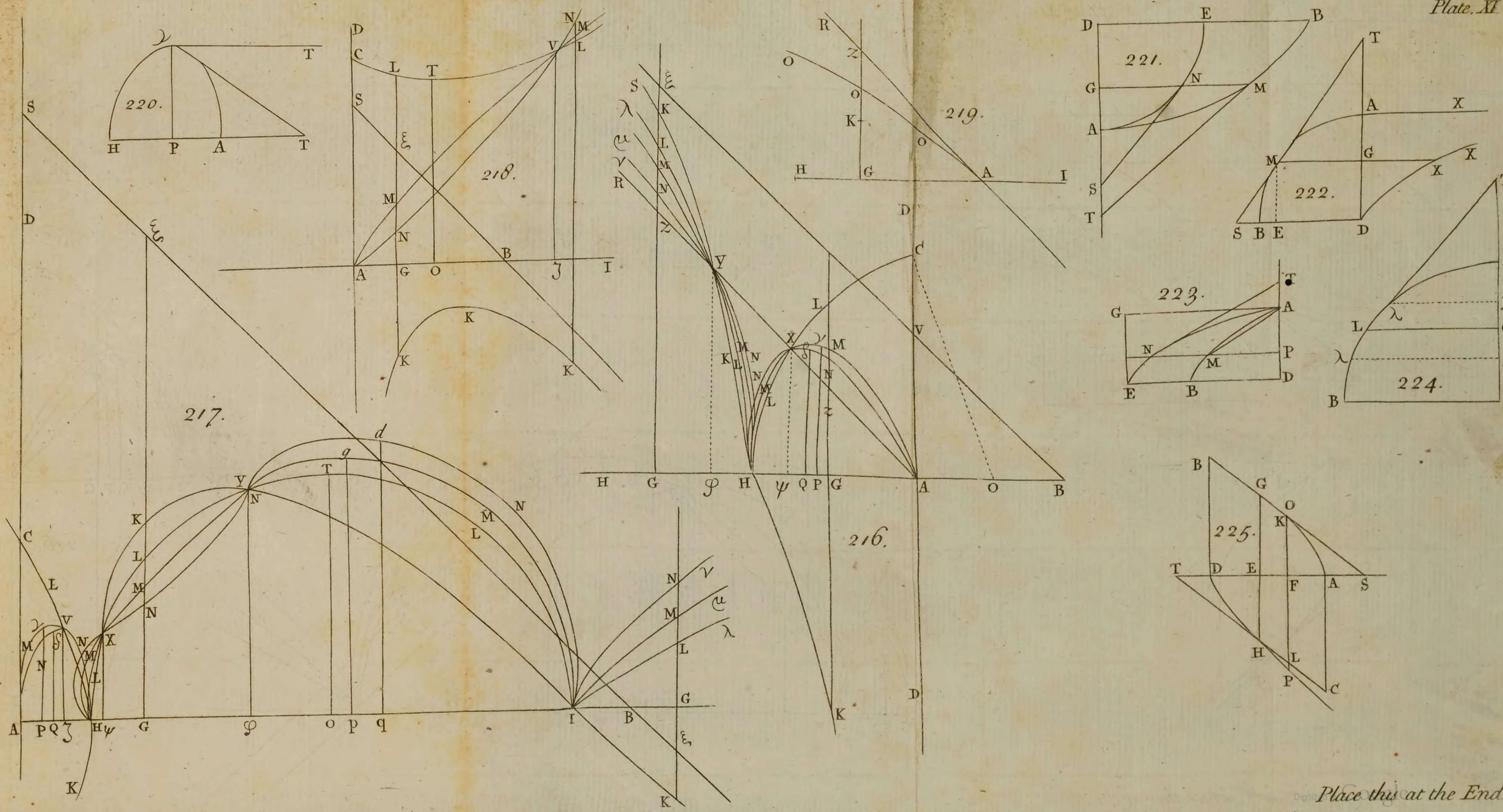
Scheme;) it ſhall be $GH = QP \sqsubset KL$.

Note, This is true when the Concave Parts of the Curves lie next to the Axis; otherwiſe GEH will be a *Minimum*.

2. If $ES = ET$, the Rectangle under EG, EH will be a *Maximum*. For becauſe $SE : SF :: EG : FO$, and $TE : TF :: EH : FP$. It ſhall be $SE \times TE : SF \times TF :: EG \times EH : FO \times FP$. Wherefore becauſe it is $SE \times TE \sqsubset SF \times TF$, it ſhall be $EG \times EH \sqsubset FO \times FP$.

FINIS.

Plate. XI



216.
217.
218.
219.
220.
221.
222.
223.
224.
225.
A B C D E G H I K L M N O P Q S T V X Y Z
φ ψ ξ λ μ ν δ ζ
d g p q o

Place this at the End

We uſe the following Notes or Characters for the Sake of Brevity.

$A + B$. That is, A and B taken together.
$A - B$. A, leſſen'd by B.
$A -: B$. Difference of the Letters A. and B.
$A \times B$. A Dividend by B.
$\frac{A}{B}$. A multiplied or drawn into B.
$A = B$. A equal to B.
$A \sqsubset B$. A is greater than B.
$A \sqsupset B$. A is leſſer than B.
$A : B :: C : D$. A has the ſame Ratio to B as C to D.
A, B, C, D ∺ A, B, C, D are continual Proportionals.
$A : B \sqsubset C : D$. A has a greater Ratio to B than C to D.
$A : B \sqsupset C : D$. A has a leſſer Ratio to B than C to D.

$A : B + C : D \left.\begin{matrix} = \\ \sqsubset \\ \sqsupset \end{matrix}\right\} M : N$. The Ratio's compounded of A to B and C to D $\left\{\begin{matrix} \text{are equal} \\ \text{exceed} \\ \text{leſs} \end{matrix}\right\}$ than the Ratio of M to N.

$A\,q$. The Square of A.
$\surd\,A$. The Side or Square Root of A.
A Cub. The Cube of A.
$\surd\,A\,q + B\,q$. The ſide Compounded of $A\,q$ and $B\,q$.

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